

A FINITE ELEMENT APPROXIMATION FOR THE STOCHASTIC MAXWELL–LANDAU–LIFSHITZ–GILBERT SYSTEM

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ABSTRACT. The stochastic Landau–Lifshitz–Gilbert (LLG) equation coupled with the Maxwell equations (the so called stochastic MLLG system) describes the creation of domain walls and vortices (fundamental objects for the novel nanostructured magnetic memories). We first reformulate the stochastic LLG equation into an equation with time-differentiable solutions. We then propose a convergent θ -linear scheme to approximate the solutions of the reformulated system. As a consequence, we prove convergence of the approximate solutions, with no or minor conditions on time and space steps (depending on the value of θ). Hence, we prove the existence of weak martingale solutions of the stochastic MLLG system. Numerical results are presented to show applicability of the method.

1. INTRODUCTION

The Maxwell–Landau–Lifshitz–Gilbert (MLLG) system describes the electromagnetic behaviour of a ferromagnetic material [12]. For simplicity, we suppose that there is a bounded cavity $\tilde{D} \subset \mathbb{R}^3$ (with perfectly conducting outer surface $\partial\tilde{D}$) in which a ferromagnet D is embedded, and $\tilde{D} \setminus \bar{D}$ is an isotropic material. Letting $D_T := (0, T) \times D$ and $\tilde{D}_T := (0, T) \times \tilde{D}$, the magnetisation field $\mathbf{M} : D_T \rightarrow \mathbb{S}^2$ (where \mathbb{S}^2 is the unit sphere in \mathbb{R}^3) and the magnetic field $\mathbf{H} : \tilde{D}_T \rightarrow \mathbb{R}^3$ satisfy the quasi-static model of the MLLG system:

$$(1.1) \quad \mathbf{M}_t = \lambda_1 \mathbf{M} \times \mathbf{H}_{\text{eff}} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_{\text{eff}}) \quad \text{in } D_T,$$

$$(1.2) \quad \mu_0 \mathbf{H}_t + \nabla \times (\sigma \nabla \times \mathbf{H}) = -\mu_0 \tilde{\mathbf{M}}_t \quad \text{in } \tilde{D}_T,$$

in which $\lambda_1 \neq 0$, $\lambda_2 > 0$, and $\mu_0 > 0$ are constants. Here, the inverse of the conductivity σ is a scalar positive bounded function on \tilde{D} satisfying $\sigma(\mathbf{x}) = \sigma_D > 0$ for all $\mathbf{x} \in D$ [24]. Vector function \mathbf{H}_{eff} is the effective field and $\tilde{\mathbf{M}} : \tilde{D}_T \rightarrow \mathbb{R}^3$ is the zero extension of \mathbf{M} onto \tilde{D}_T , i.e.,

$$\tilde{\mathbf{M}}(t, \mathbf{x}) = \begin{cases} \mathbf{M}(t, \mathbf{x}), & (t, \mathbf{x}) \in D_T, \\ 0, & (t, \mathbf{x}) \in \tilde{D}_T \setminus \bar{D}_T. \end{cases}$$

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The system (1.1)–(1.2) is supplemented with the initial conditions

$$(1.3) \quad \mathbf{M}(0, \cdot) = \mathbf{M}_0 \text{ in } D \quad \text{and} \quad \mathbf{H}(0, \cdot) = \mathbf{H}_0 \text{ in } \tilde{D},$$

and the boundary conditions

$$(1.4) \quad \partial_{\mathbf{n}_D} \mathbf{M} = 0 \text{ on } (0, T) \times \partial D \quad \text{and} \quad (\nabla \times \mathbf{H}) \times \mathbf{n}_{\tilde{D}} = 0 \text{ on } (0, T) \times \partial \tilde{D},$$

where \mathbf{n}_D and $\mathbf{n}_{\tilde{D}}$ are the unit outward normal vectors to D and \tilde{D} , respectively. Here $\partial_{\mathbf{n}_D}$ denotes the normal derivative.

It is highly significant to consider the stochastic MLLG system in order to describe the creation of domain walls and vortices (fundamental objects for the novel nanostructured magnetic memories) [26]. We follow [6, 9] to add a noise to the effective field \mathbf{H}_{eff} so that the stochastic version of the MLLG system takes the form

$$(1.5) \quad d\mathbf{M} = (\lambda_1 \mathbf{M} \times \mathbf{H}_{\text{eff}} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_{\text{eff}})) dt + (\mathbf{M} \times \mathbf{g}) \circ dW(t) \text{ in } D_T,$$

$$(1.6) \quad \mu_0 d\mathbf{H} + \nabla \times (\sigma \nabla \times \mathbf{H}) dt = -\mu_0 d\tilde{\mathbf{M}} \text{ in } \tilde{D}_T,$$

where $\mathbf{g} : D \rightarrow \mathbb{R}^3$ is a given bounded function, and W is a one-dimensional Wiener process. Here $\circ dW(t)$ stands for the Stratonovich differential. We assume without loss of generality that (see [9])

$$(1.7) \quad |\mathbf{g}(\mathbf{x})| = 1, \quad \mathbf{x} \in D$$

For simplicity the effective field \mathbf{H}_{eff} is taken to be $\mathbf{H}_{\text{eff}} = \Delta \mathbf{M} + \mathbf{H}$.

In the deterministic case, i.e. (1.1)–(1.2), the existence and uniqueness of a *local* strong solution is shown by Cimrák [11]. He also proposes [10] a finite element method to approximate this local solution and provides error estimation. Various results on the existence of global weak solutions are proved in [17, 18, 27]. A more complete list can be found in [12, 16, 20]. It should be noted that apart from [10] where a numerical scheme is suggested for a local solution, other analyses are non-constructive, namely no computational techniques are proposed for the solution.

In [25], the stability of a semidiscrete scheme to numerically solve (1.1)–(1.2) is verified, but its convergence is not studied. Bañas, Bartels and Prohl [4] propose an implicit *nonlinear* scheme to solve the MLLG system, and succeed in proving that the finite element solution converges to a weak *global* solution of the problem. A *θ -linear* finite element scheme is proposed in [7, 21, 22] to find a weak *global* solution to the MLLG system, and convergence of the numerical solutions is proved with no condition imposed on time step and space step if $\theta \in (\frac{1}{2}, 1]$. It should be mentioned that the proofs of existence proposed in [4, 7, 21, 22] are constructive proofs, namely an approximate solution can be computed.

In the stochastic case, the Faedo–Galerkin method is used in [9] to show the existence of a weak martingale solution for the stochastic Landau–Lifshitz–Gilbert (LLG) equation (1.5). Finite element schemes for this equation are studied in [2, 6, 14] which prove that the numerical solutions converge to a weak martingale solution. It is noted that a *non-linear* scheme is proposed in [6] and *linear* schemes are proposed in [2, 14].

The full version of the stochastic Landau–Lifshitz equation coupled with the Maxwell’s equations is studied firstly in [23, Section 5] where the existence of the weak martingale solution and its regularity are proved by using the Faedo-Galerkin approximation, the methods of compactness and Skorokhod’s Theorem.

To the best of our knowledge the numerical analysis of the system (1.5)–(1.6) is an open problem at present. In this paper, we extend the θ -linear finite element scheme developed in [22] for the deterministic MLLG system to the stochastic case. Since this scheme seeks to approximate the time derivative of the magnetization \mathbf{M} , we adopt the technique in [14] to reformulate system (1.5)–(1.6) into a system not involving the Stratonovich differential $\circ dW(t)$. Then the θ -linear scheme mentioned above can be applied. As a consequence, we prove the existence of weak martingale solutions to the stochastic MLLG system.

The paper is organised as follows. In Section 2 we define the notations to be used, and recall some technical results. In Section 3 we define weak martingale solutions to (1.5)–(1.6) and state our main result. Details of the reformulation of (1.5) are presented in Section 4. We also show in this section how a weak solution to (1.5)–(1.6) can be obtained from a weak solution of the reformulated system. In Section 5, we introduce our finite element scheme and present a proof of the convergence of finite element solutions to a weak solution of the reformulated system. Section 6 is devoted to the proof of the main theorem. Our numerical experiments are presented in Section 7.

Throughout this paper, c denotes a generic constant which may take different values at different occurrences.

2. NOTATIONS AND TECHNICAL RESULTS

2.1. Notations. In this subsection, we introduce some function spaces and notations which are used in the rest of this paper.

For any open set $U \subset \mathbb{R}^3$, the *curl* operator of a vector function $\mathbf{u} = (u_1, u_2, u_3)$ defined on U is denoted by

$$\operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u} := \left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right),$$

if the partial derivatives exist. The function spaces $\mathbb{H}^1(U)$ and $\mathbb{H}(\operatorname{curl}; U)$ are defined, respectively, by

$$\begin{aligned} \mathbb{H}^1(U) &= \left\{ \mathbf{u} \in \mathbb{L}^2(U) : \frac{\partial u_i}{\partial x_i} \in \mathbb{L}^2(U) \text{ for } i = 1, 2, 3 \right\}, \\ \mathbb{H}(\operatorname{curl}; U) &= \{ \mathbf{u} \in \mathbb{L}^2(U) : \nabla \times \mathbf{u} \in \mathbb{L}^2(U) \}. \end{aligned}$$

Here, $\mathbb{L}^2(U)$ is the usual space of Lebesgue square integrable functions defined on U and taking values in \mathbb{R}^3 . The inner product and norm in $\mathbb{L}^2(U)$ are denoted by $\langle \cdot, \cdot \rangle_U$ and $\| \cdot \|_U$, respectively.

For any vector functions $\mathbf{u}, \mathbf{v}, \mathbf{w}$, we denote

$$\begin{aligned}
 \nabla \mathbf{u} \cdot \nabla \mathbf{v} &:= \sum_{i=1}^3 \frac{\partial \mathbf{u}}{\partial x_i} \cdot \frac{\partial \mathbf{v}}{\partial x_i}, \\
 \nabla \mathbf{u} \times \nabla \mathbf{v} &:= \sum_{i=1}^3 \frac{\partial \mathbf{u}}{\partial x_i} \times \frac{\partial \mathbf{v}}{\partial x_i}, \\
 \mathbf{u} \times \nabla \mathbf{v} &:= \sum_{i=1}^3 \mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x_i}, \\
 (\mathbf{u} \times \nabla \mathbf{v}) \cdot \nabla \mathbf{w} &:= \sum_{i=1}^3 \left(\mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x_i} \right) \cdot \frac{\partial \mathbf{w}}{\partial x_i},
 \end{aligned}
 \tag{2.1}$$

provided that the partial derivatives exist, at least in the weak sense. We also denote

$$\begin{aligned}
 \mathbb{C}^\infty(\tilde{D}) &:= \left\{ \mathbf{u} : \tilde{D} \rightarrow \mathbb{R}^3 \mid \mathbf{u} \text{ is infinitely differentiable} \right\}, \\
 \mathbb{C}_\times^\infty(\tilde{D}) &:= \left\{ \mathbf{u} \in \mathbb{C}^\infty(\tilde{D}) \cap \mathbb{C}(\bar{D}) \mid (\nabla \times \mathbf{u}) \times \mathbf{n}_D = 0 \text{ on } \partial D \text{ and } (\nabla \times \mathbf{u}) \times \mathbf{n}_{\tilde{D}} = 0 \text{ on } \partial \tilde{D} \right\}, \\
 C_T^1(0, T; E) &:= \{ \mathbf{u} : [0, T] \rightarrow E \mid \mathbf{u} \text{ is continuously differentiable and } \mathbf{u}(T) = 0 \text{ in } E \}, \\
 C_0^1(0, T; E) &:= \{ \mathbf{u} : [0, T] \rightarrow E \mid \mathbf{u} \text{ is continuously differentiable and } \mathbf{u}(0) = \mathbf{u}(T) = 0 \text{ in } E \},
 \end{aligned}$$

for any $T > 0$ and any normed vector space E .

2.2. Technical results. In this subsection we recall some results from [14]. They will be used in the next section to reformulate (3.1) to a new form.

Assume that $\mathbf{g} \in \mathbb{L}^\infty(D)$, and let $G : \mathbb{L}^2(D) \rightarrow \mathbb{L}^2(D)$ be defined by

$$G\mathbf{u} = \mathbf{u} \times \mathbf{g} \quad \forall \mathbf{u} \in \mathbb{L}^2(D). \tag{2.2}$$

Then the operator G is bounded [14].

Lemma 2.1. *For any $s \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{L}^2(D)$ there hold*

$$e^{sG}\mathbf{u} = \mathbf{u} + (\sin s)G\mathbf{u} + (1 - \cos s)G^2\mathbf{u}, \tag{2.3}$$

$$(e^{sG})^* = e^{-sG}, \tag{2.4}$$

$$e^{sG}G\mathbf{u} = Ge^{sG}\mathbf{u}, \tag{2.5}$$

$$e^{sG}(\mathbf{u} \times \mathbf{v}) = e^{sG}\mathbf{u} \times e^{sG}\mathbf{v}. \tag{2.6}$$

In the proof of the existence of weak solutions we also need the following result for the operator e^{sG} .

Lemma 2.2. *Assume that $\mathbf{g} \in \mathbb{H}^2(D)$. For any $s \in \mathbb{R}$, $\mathbf{u} \in \mathbb{H}^1(D)$ and $\mathbf{v} \in \mathbb{W}_0^{1,\infty}(D)$, let*

$$\tilde{C}(s, \mathbf{v}) = e^{-sG}((\sin s)C + (1 - \cos s)(GC + CG))\mathbf{v}$$

with C being defined by

$$C\mathbf{u} = \mathbf{u} \times \Delta \mathbf{g} + 2 \sum_{i=1}^3 \frac{\partial \mathbf{u}}{\partial x_i} \times \frac{\partial \mathbf{g}}{\partial x_i}.$$

There holds

$$\left\langle \tilde{C}(s, e^{-sG}\mathbf{u}), \mathbf{v} \right\rangle_D = \langle \nabla(e^{-sG}\mathbf{u}), \nabla \mathbf{v} \rangle_D - \langle \nabla \mathbf{u}, \nabla(e^{sG}\mathbf{v}) \rangle_D,$$

From now on, we assume that $\mathbf{g} \in \mathbb{W}^{2,\infty}(D)$.

We finish this section by stating two elementary identities involving the dot and cross products of vectors in \mathbb{R}^3 , which will be frequently used. For all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, the following identities hold

$$(2.7) \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

and

$$(2.8) \quad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}.$$

3. THE MAIN RESULT

In this section we state the definition of a weak martingale solution to (1.5)–(1.6) and our main result.

Recalling that $\mathbf{H}_{\text{eff}} = \Delta \mathbf{M} + \mathbf{H}$, multiplying (1.5) by a test function $\psi \in \mathbb{C}^\infty(D)$ and integrating over $(0, t) \times D$ we obtain formally

$$\begin{aligned} \langle \mathbf{M}(t), \psi \rangle_D - \langle \mathbf{M}_0, \psi \rangle_D &= \lambda_1 \int_0^t \langle \mathbf{M} \times \Delta \mathbf{M}, \psi \rangle_D ds + \lambda_1 \int_0^t \langle \mathbf{M} \times \mathbf{H}, \psi \rangle_D ds \\ &\quad - \lambda_2 \int_0^t \langle \mathbf{M} \times (\mathbf{M} \times \Delta \mathbf{M}), \psi \rangle_D ds \\ &\quad - \lambda_2 \int_0^t \langle \mathbf{M} \times (\mathbf{M} \times \mathbf{H}), \psi \rangle_D ds \\ &\quad + \int_0^t \langle \mathbf{M} \times \mathbf{g}, \psi \rangle_D \circ dW. \end{aligned}$$

From (2.8), the Green identity and $\nabla \mathbf{M} \cdot (\nabla \mathbf{M} \times \psi) = 0$ we define

$$\begin{aligned} \langle \mathbf{M} \times \Delta \mathbf{M}, \psi \rangle_D &= -\langle \Delta \mathbf{M}, \mathbf{M} \times \psi \rangle_D \\ &:= \langle \nabla \mathbf{M}, \nabla(\mathbf{M} \times \psi) \rangle_D \\ &= \langle \nabla \mathbf{M}, \nabla \mathbf{M} \times \psi \rangle_D + \langle \nabla \mathbf{M}, \mathbf{M} \times \nabla \psi \rangle_D \\ &= -\langle \mathbf{M} \times \nabla \mathbf{M}, \nabla \psi \rangle_D, \end{aligned}$$

and similarly

$$\langle \mathbf{M} \times (\mathbf{M} \times \Delta \mathbf{M}), \psi \rangle_D := \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla(\mathbf{M} \times \psi) \rangle_D.$$

Therefore,

$$\begin{aligned} \langle \mathbf{M}(t), \boldsymbol{\psi} \rangle_D - \langle \mathbf{M}_0, \boldsymbol{\psi} \rangle_D &= -\lambda_1 \int_0^t \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla \boldsymbol{\psi} \rangle_D ds + \lambda_1 \int_0^t \langle \mathbf{M} \times \mathbf{H}, \boldsymbol{\psi} \rangle_D ds \\ &\quad - \lambda_2 \int_0^t \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla (\mathbf{M} \times \boldsymbol{\psi}) \rangle_D ds \\ &\quad - \lambda_2 \int_0^t \langle \mathbf{M} \times (\mathbf{M} \times \mathbf{H}), \boldsymbol{\psi} \rangle_D ds + \int_0^t \langle \mathbf{M} \times \mathbf{g}, \boldsymbol{\psi} \rangle_D \circ dW(s). \end{aligned}$$

In the same manner, if we multiply (1.6) by a test function $\boldsymbol{\zeta} \in C_T^1(0, T; \mathbb{C}_\times^\infty(\tilde{D}))$, integrate over \tilde{D}_T , and note (1.3), then we obtain, formally,

$$\mu_0 \left\langle \mathbf{H} + \widetilde{\mathbf{M}}, \boldsymbol{\zeta}_t \right\rangle_{\tilde{D}_T} - \mu_0 \left\langle \mathbf{H}_0 + \widetilde{\mathbf{M}}_0, \boldsymbol{\zeta}(0) \right\rangle_{\tilde{D}} = \langle \nabla \times (\sigma \nabla \times \mathbf{H}), \boldsymbol{\zeta} \rangle_{\tilde{D}}.$$

We remark that the time derivative is taken on $\boldsymbol{\zeta}$ because in general $\widetilde{\mathbf{M}}$ is not time differentiable. Since $(\nabla \times \mathbf{H}) \times \mathbf{n}_{\tilde{D}} = 0$, see (1.4), and $(\nabla \times \boldsymbol{\zeta}) \times \mathbf{n}_{\tilde{D}} = 0$, see the definition of $\mathbb{C}_\times^\infty(\tilde{D})$ in Section 2, it follows from [24, Corollary 3.20] that

$$\langle \nabla \times (\sigma \nabla \times \mathbf{H}), \boldsymbol{\zeta} \rangle_{\tilde{D}} = \langle \sigma \nabla \times \mathbf{H}, \nabla \times \boldsymbol{\zeta} \rangle_{\tilde{D}}.$$

Hence

$$\mu_0 \left\langle \mathbf{H} + \widetilde{\mathbf{M}}, \boldsymbol{\zeta}_t \right\rangle_{\tilde{D}_T} - \mu_0 \left\langle \mathbf{H}_0 + \widetilde{\mathbf{M}}_0, \boldsymbol{\zeta}(0) \right\rangle_{\tilde{D}} = \langle \sigma \nabla \times \mathbf{H}, \nabla \times \boldsymbol{\zeta} \rangle_{\tilde{D}_T}.$$

The above observations prompt us to define the solution of (1.5)–(1.6) as follows.

Definition 3.1. *Given $T \in (0, \infty)$, a weak martingale solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}, W, \mathbf{M}, \mathbf{H})$ to (1.5)–(1.6) on the time interval $[0, T]$, consists of*

- (a) *a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ with the filtration satisfying the usual conditions,*
- (b) *a one-dimensional (\mathcal{F}_t) -adapted Wiener process $W = (W_t)_{t \in [0, T]}$,*
- (c) *a progressively measurable process $\mathbf{M} : [0, T] \times \Omega \rightarrow \mathbb{L}^2(D)$,*
- (d) *a progressively measurable process $\mathbf{H} : [0, T] \times \Omega \rightarrow \mathbb{L}^2(\tilde{D})$*

such that there hold

- (1) $\mathbb{P}(\mathbf{M} \in C([0, T]; \mathbb{H}^{-1}(D))) = 1;$
- (2) $\mathbb{P}(\mathbf{H} \in L^2(0, T; \mathbb{H}(\text{curl}; \tilde{D}))) = 1;$
- (3) $\mathbb{E}(\text{ess sup}_{t \in [0, T]} \|\nabla \mathbf{M}(t)\|_D^2) < \infty;$
- (4) *for all $t \in [0, T]$, $|\mathbf{M}(t, \cdot)| = 1$ a.e. in D , and \mathbb{P} -a.s.;*

(5) for every $t \in [0, T]$, for all $\psi \in \mathbb{C}^\infty(D)$, \mathbb{P} -a.s.:

$$\begin{aligned}
 \langle \mathbf{M}(t), \psi \rangle_D - \langle \mathbf{M}_0, \psi \rangle_D &= -\lambda_1 \int_0^t \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla \psi \rangle_D ds \\
 &\quad - \lambda_2 \int_0^t \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla (\mathbf{M} \times \psi) \rangle_D ds \\
 &\quad + \lambda_1 \int_0^t \langle \mathbf{M} \times \mathbf{H}, \psi \rangle_D ds \\
 &\quad - \lambda_2 \int_0^t \langle \mathbf{M} \times (\mathbf{M} \times \mathbf{H}), \psi \rangle_D ds \\
 &\quad + \int_0^t \langle \mathbf{M} \times \mathbf{g}, \psi \rangle_D \circ dW(s);
 \end{aligned}
 \tag{3.1}$$

(6) for all $\zeta \in C_T^1(0, T; \mathbb{C}_\infty^\infty(\tilde{D}))$, \mathbb{P} -a.s.:

$$\mu_0 \left\langle \mathbf{H} + \widetilde{\mathbf{M}}, \zeta_t \right\rangle_{\tilde{D}_T} - \mu_0 \left\langle \mathbf{H}_0 + \widetilde{\mathbf{M}}_0, \zeta(0, \cdot) \right\rangle_{\tilde{D}} = \langle \sigma \nabla \times \mathbf{H}, \nabla \times \zeta \rangle_{\tilde{D}_T}.
 \tag{3.2}$$

The main theorem of the paper is stated below.

Theorem 3.2. Assume that $\mathbf{g} \in \mathbb{W}^{2,\infty}(D)$ satisfies (1.7) and $(\mathbf{M}_0, \mathbf{H}_0)$ satisfies

$$\begin{aligned}
 \mathbf{M}_0 &\in \mathbb{H}^2(D), \quad |\mathbf{M}_0| = 1 \text{ a.e. in } D, \\
 (\mathbf{H}_0 + \widetilde{\mathbf{M}}_0) &\in \mathbb{H}^1(\tilde{D}), \quad \nabla \times (\mathbf{H}_0 + \widetilde{\mathbf{M}}_0) \in \mathbb{H}^1(\tilde{D}).
 \end{aligned}
 \tag{3.3}$$

For each $T > 0$, there exists a weak martingale solution to (1.5)–(1.6).

Proof. The theorem is a direct consequence of Theorem 6.9. \square

4. EQUIVALENCE OF WEAK SOLUTIONS

In this section, we use the operator G defined in Section 2 to define new variables \mathbf{m} and \mathbf{P} from \mathbf{M} and \mathbf{H} .

Informally, if (\mathbf{M}, \mathbf{H}) is a weak solution to (3.1)–(3.2) then we can define new processes \mathbf{m} and \mathbf{P} (see (4.1)–(4.2) below) such that the Stratonovich differential $\circ dW(t)$ vanishes in the partial differential equation satisfied by \mathbf{m} . Moreover, it will be seen that \mathbf{m} is differentiable with respect to t . We will make this argument more rigorous in the following lemma.

Let a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ and a Wiener process $W(t)$ on it be given. We define a new processes \mathbf{m} and \mathbf{P} from processes \mathbf{M} and \mathbf{H}

$$\mathbf{m}(t, \cdot) := e^{-W(t)G} \mathbf{M}(t, \cdot) \quad \forall t \geq 0, \text{ a.e. in } D,
 \tag{4.1}$$

$$\mathbf{P}(t, \cdot) := \mathbf{H}(t, \cdot) + \widetilde{\mathbf{M}}(t, \cdot) \quad \forall t \geq 0, \text{ a.e. in } \tilde{D},
 \tag{4.2}$$

$$\mathbf{P}_0 := \mathbf{H}_0 + \widetilde{\mathbf{M}}_0 \quad \text{a.e. in } \tilde{D},$$

where $\widetilde{\mathbf{M}}_0$ is the zero extension of \mathbf{M}_0 onto \widetilde{D} . Then it follows immediately from (??) and (2.4) that, for all $t \in [0, T]$ and almost all $x \in D$,

$$(4.3) \quad |\mathbf{M}(t, \cdot)| = 1 \quad \text{if and only if} \quad |\mathbf{m}(t, \cdot)| = 1.$$

The following lemma shows that in order to find \mathbf{M} and \mathbf{H} , it suffices to find \mathbf{m} and \mathbf{P} .

Lemma 4.1. *Let $\mathbf{m} \in H^1(0, T; \mathbb{H}^1(D))$ and $\mathbf{P} \in L^2(0, T; \mathbb{H}(\text{curl}; \widetilde{D}))$, \mathbb{P} -a.s., satisfy*

$$(4.4) \quad \begin{aligned} & \langle \mathbf{m}_t, \boldsymbol{\xi} \rangle_{D_T} + \lambda_1 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla \boldsymbol{\xi} \rangle_{D_T} + \lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\mathbf{m} \times \boldsymbol{\xi}) \rangle_{D_T} - \langle F(t, \mathbf{m}), \boldsymbol{\xi} \rangle_{D_T} \\ & - \lambda_1 \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, \boldsymbol{\xi} \rangle_{D_T} + \lambda_2 \langle \mathbf{m} \times (\mathbf{m} \times e^{-W(t)G} \mathbf{P}), \boldsymbol{\xi} \rangle_{D_T} = 0 \end{aligned}$$

and

$$(4.5) \quad \mu_0 \langle \mathbf{P}, \boldsymbol{\zeta}_t \rangle_{\widetilde{D}_T} - \mu_0 \langle \mathbf{P}_0, \boldsymbol{\zeta}(0, \cdot) \rangle_{\widetilde{D}} = \langle \sigma \nabla \times \mathbf{P}, \nabla \times \boldsymbol{\zeta} \rangle_{\widetilde{D}_T} - \langle \sigma \nabla \times (e^{W(t)G} \mathbf{m}), \nabla \times \boldsymbol{\zeta} \rangle_{D_T},$$

where

$$(4.6) \quad F(t, \mathbf{m}) = \lambda_1 \mathbf{m} \times \widetilde{C}(W(t), \mathbf{m}) - \lambda_2 \mathbf{m} \times (\mathbf{m} \times \widetilde{C}(W(t), \mathbf{m}))$$

for all $\boldsymbol{\xi} \in L^2(0, T; \mathbb{W}^{1,\infty}(D))$ and $\boldsymbol{\zeta} \in C_T^1(0, T; \mathbb{C}_\times^\infty(\widetilde{D}))$, with \widetilde{C} defined in Lemma 2.2. Then $\mathbf{M} = e^{W(t)G} \mathbf{m}$ and $\mathbf{H} = \mathbf{P} - \widetilde{\mathbf{M}}$ satisfy (3.1)–(3.2) \mathbb{P} -a.s.

Proof.

Step 1: \mathbf{M} and \mathbf{H} satisfy (3.1):

Since $e^{W(t)G}$ is a semimartingale and \mathbf{m} is absolutely continuous, using Itô's formula for $\mathbf{M} = e^{W(t)G} \mathbf{m}$ (see e.g. [13]), we deduce

$$(4.7) \quad \begin{aligned} \mathbf{M}(t) &= \mathbf{M}(0) + \int_0^t G e^{W(s)G} \mathbf{m} dW(s) + \int_0^t \frac{1}{2} G^2 e^{W(s)G} \mathbf{m} ds + \int_0^t e^{W(s)G} \mathbf{m}_t ds \\ &= \mathbf{M}(0) + \int_0^t G \mathbf{M} dW(s) + \frac{1}{2} \int_0^t G^2 \mathbf{M} ds + \int_0^t e^{W(s)G} \mathbf{m}_t ds, \end{aligned}$$

where the first integral on the right-hand side is an Itô integral and the last two are Bochner integrals. Recalling the relation between the Stratonovich and Itô differentials, namely

$$(4.8) \quad (G\mathbf{u}) \circ dW(s) = G\mathbf{u} dW(s) + \frac{1}{2} G'(\mathbf{u})[G\mathbf{u}] ds,$$

and noting that $G'(\mathbf{u})[G\mathbf{u}] = G^2 \mathbf{u}$, we rewrite (4.7) in the Stratonovich form as

$$\mathbf{M}(t) = \mathbf{M}(0) + \int_0^t G \mathbf{M} \circ dW(s) + \int_0^t e^{W(s)G} \mathbf{m}_t ds.$$

Multiplying both sides of the above equation by a test function $\psi \in \mathbb{C}^\infty(D)$ and integrating over D we obtain

$$\begin{aligned}
 \langle \mathbf{M}(t), \psi \rangle_D &= \langle \mathbf{M}(0), \psi \rangle_D + \int_0^t \langle G\mathbf{M}, \psi \rangle_D \circ dW(s) + \int_0^t \langle e^{W(s)G} \mathbf{m}_t, \psi \rangle_D ds \\
 (4.9) \quad &= \langle \mathbf{M}(0), \psi \rangle_D + \int_0^t \langle G\mathbf{M}, \psi \rangle_D \circ dW(s) + \int_0^t \langle \mathbf{m}_t, e^{-W(s)G} \psi \rangle_D ds,
 \end{aligned}$$

where in the last step we used (2.4).

On the other hand, we note that $e^{-W(\cdot)G} \psi \in L^2(0, t; \mathbb{W}^{1,\infty}(D))$ for $t \in [0, T]$. Let the test function ξ in (4.4) be $e^{-W(\cdot)G} \psi$, we obtain from (4.6) that

$$\begin{aligned}
 \int_0^t \langle \mathbf{m}_t, e^{-W(s)G} \psi \rangle_D ds &= -\lambda_1 \int_0^t \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (e^{-W(s)G} \psi) \rangle_D ds \\
 &\quad - \lambda_2 \int_0^t \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\mathbf{m} \times (e^{-W(s)G} \psi)) \rangle_D ds \\
 &\quad + \lambda_1 \int_0^t \langle \mathbf{m} \times \tilde{C}(W(s), \mathbf{m}), e^{-W(s)G} \psi \rangle_D ds \\
 &\quad - \lambda_2 \int_0^t \langle \mathbf{m} \times (\mathbf{m} \times \tilde{C}(W(s), \mathbf{m})), e^{-W(s)G} \psi \rangle_D ds \\
 &\quad + \lambda_1 \int_0^t \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, e^{-W(s)G} \psi \rangle_D ds \\
 &\quad - \lambda_2 \int_0^t \langle \mathbf{m} \times (\mathbf{m} \times e^{-W(t)G} \mathbf{P}), e^{-W(s)G} \psi \rangle_D ds \\
 &=: \int_0^t (T_1(s) + \dots + T_6(s)) ds.
 \end{aligned}$$

Considering T_3 , we use successively (2.8), Lemma 2.2, (4.1), and (2.6) to obtain

$$\begin{aligned}
 T_3(s) &= \lambda_1 \langle \mathbf{m} \times \tilde{C}(W(s), \mathbf{m}), e^{-W(s)G} \psi \rangle_D = -\lambda_1 \langle \tilde{C}(W(s), \mathbf{m}), \mathbf{m} \times e^{-W(s)G} \psi \rangle_D \\
 &= -\lambda_1 \langle \nabla \mathbf{m}, \nabla (\mathbf{m} \times e^{-W(s)G} \psi) \rangle_D + \lambda_1 \langle \nabla \mathbf{M}, \nabla (\mathbf{M} \times \psi) \rangle_D \\
 &= -\lambda_1 \langle \nabla \mathbf{m}, \mathbf{m} \times \nabla (e^{-W(s)G} \psi) \rangle_D + \lambda_1 \langle \nabla \mathbf{M}, \mathbf{M} \times \nabla \psi \rangle_D \\
 &= \lambda_1 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (e^{-W(s)G} \psi) \rangle_D - \lambda_1 \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla \psi \rangle_D.
 \end{aligned}$$

Therefore,

$$T_1 + T_3 = -\lambda_1 \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla \psi \rangle_D.$$

Similarly, considering T_4 we have

$$\begin{aligned}
 T_4(s) &= -\lambda_2 \langle \mathbf{m} \times (\mathbf{m} \times \tilde{C}(W(s), \mathbf{m})), e^{-W(s)G} \psi \rangle_D \\
 &= \lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\mathbf{m} \times e^{-W(s)G} \psi) \rangle_D - \lambda_2 \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla (\mathbf{M} \times \psi) \rangle_D,
 \end{aligned}$$

so that

$$T_2 + T_4 = -\lambda_2 \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla (\mathbf{M} \times \psi) \rangle_D.$$

On the other hand, by using (2.4), (2.6), and noting that $\mathbf{P} = \mathbf{H} + \mathbf{M}$ in D , we obtain

$$T_5(s) = \lambda_1 \langle \mathbf{m} \times e^{-W(s)G} \mathbf{P}, e^{-W(s)G} \boldsymbol{\psi} \rangle_D = \lambda_1 \langle \mathbf{M} \times \mathbf{H}, \boldsymbol{\psi} \rangle_D$$

and

$$T_6(s) = -\lambda_2 \langle \mathbf{m} \times (\mathbf{m} \times e^{-W(s)G} \mathbf{P}), e^{-W(s)G} \boldsymbol{\psi} \rangle_D = -\lambda_2 \langle \mathbf{M} \times (\mathbf{M} \times \mathbf{H}), \boldsymbol{\psi} \rangle_D.$$

Therefore,

$$\begin{aligned} \int_0^t \langle \mathbf{m}_t, e^{-W(s)G} \boldsymbol{\psi} \rangle_D ds &= -\lambda_1 \int_0^t \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla \boldsymbol{\psi} \rangle_D ds - \lambda_2 \int_0^t \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla (\mathbf{M} \times \boldsymbol{\psi}) \rangle_D ds \\ &\quad + \lambda_1 \int_0^t \langle \mathbf{M} \times \mathbf{H}, \boldsymbol{\psi} \rangle_D ds - \lambda_2 \int_0^t \langle \mathbf{M} \times (\mathbf{M} \times \mathbf{H}), \boldsymbol{\psi} \rangle_D ds. \end{aligned}$$

This equation and (4.9) give

$$\begin{aligned} \langle \mathbf{M}(t), \boldsymbol{\psi} \rangle_D &= \langle \mathbf{M}(0), \boldsymbol{\psi} \rangle_D + \int_0^t \langle G\mathbf{M}, \boldsymbol{\psi} \rangle_D \circ dW(s) \\ &\quad - \lambda_1 \int_0^t \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla \boldsymbol{\psi} \rangle_D ds - \lambda_2 \int_0^t \langle \mathbf{M} \times \nabla \mathbf{M}, \nabla (\mathbf{M} \times \boldsymbol{\psi}) \rangle_D ds \\ &\quad + \lambda_1 \int_0^t \langle \mathbf{M} \times \mathbf{H}, \boldsymbol{\psi} \rangle_D ds - \lambda_2 \int_0^t \langle \mathbf{M} \times (\mathbf{M} \times \mathbf{H}), \boldsymbol{\psi} \rangle_D ds. \end{aligned}$$

Hence, \mathbf{M} and \mathbf{H} satisfy (3.1).

Step 2: \mathbf{M} and \mathbf{H} satisfy (3.2):

This follows immediately from (4.5) and the fact that

$$\left\langle \sigma \nabla \times \widetilde{\mathbf{M}}, \nabla \times \boldsymbol{\zeta} \right\rangle_{\widetilde{D}_T} = \langle \sigma \nabla \times \mathbf{M}, \nabla \times \boldsymbol{\zeta} \rangle_{D_T},$$

completing the proof of the lemma. \square

In the next lemma we provide an equivalence of equation (4.4), namely its Gilbert form.

Lemma 4.2. *Assume that $\mathbf{m} \in H^1(0, T; \mathbb{H}^1(D))$ and $\mathbf{P} \in \mathbb{L}^2(\widetilde{D}_T)$, \mathbb{P} -a.s., satisfy*

$$(4.10) \quad |\mathbf{m}(t, \cdot)| = 1, \quad t \in (0, T), \text{ a.e. in } D, \mathbb{P}\text{-a.s.}$$

Assume further that (\mathbf{m}, \mathbf{P}) satisfies \mathbb{P} -a.s.

$$(4.11) \quad \lambda_1 \langle \mathbf{m} \times \mathbf{m}_t, \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T} - \lambda_2 \langle \mathbf{m}_t, \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T} - \mu \langle \nabla \mathbf{m}, \mathbf{m} \times \nabla \boldsymbol{\varphi} \rangle_{D_T} \\ - \langle R(t, \mathbf{m}), \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T} + \mu \langle e^{-W(t)G} \mathbf{P}, \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T} = 0,$$

for all $\boldsymbol{\varphi} \in L^2(0, T; \mathbb{H}^1(D))$, where $\mu = \lambda_1^2 + \lambda_2^2$ and

$$R(t, \mathbf{m}) = -\lambda_1^2 \widetilde{C}(W(t), \mathbf{m}) + \lambda_2^2 \mathbf{m} \times (\mathbf{m} \times \widetilde{C}(W(t), \mathbf{m})),$$

with \widetilde{C} defined in Lemma 2.2. Then (\mathbf{m}, \mathbf{P}) satisfies (4.4) \mathbb{P} -a.s.

Proof. Firstly, we observe that for each $\boldsymbol{\xi} \in L^2(0, T; \mathbb{W}^{1,\infty}(D))$, due to Lemma 8.1, there exists $\boldsymbol{\varphi} \in L^2(0, T; \mathbb{H}^1(D))$ satisfying

$$(4.12) \quad \boldsymbol{\xi} = \lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{m}.$$

Next we derive some identities which will be used later in the proof. By using (2.7) and noting (4.10) (so that $\boldsymbol{m} \cdot \boldsymbol{m}_t = 0$), we have

$$(4.13) \quad \boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{m}_t) = -\boldsymbol{m}_t.$$

Moreover,

$$\boldsymbol{m} \times (\boldsymbol{\varphi} \times \boldsymbol{m}) = \boldsymbol{\varphi} - (\boldsymbol{m} \cdot \boldsymbol{\varphi})\boldsymbol{m} \quad \text{and} \quad \nabla(\boldsymbol{m} \times (\boldsymbol{\varphi} \times \boldsymbol{m})) = \nabla \boldsymbol{\varphi} - \nabla((\boldsymbol{m} \cdot \boldsymbol{\varphi})\boldsymbol{m}).$$

The above identities and (2.1) imply

$$(4.14) \quad (\boldsymbol{m} \times e^{-W(t)G} \boldsymbol{P}) \cdot (\boldsymbol{m} \times (\boldsymbol{\varphi} \times \boldsymbol{m})) = (\boldsymbol{m} \times e^{-W(t)G} \boldsymbol{P}) \cdot \boldsymbol{\varphi}$$

and

$$(4.15) \quad \begin{aligned} (\boldsymbol{m} \times \nabla \boldsymbol{m}) \cdot \nabla(\boldsymbol{m} \times (\boldsymbol{\varphi} \times \boldsymbol{m})) &= (\boldsymbol{m} \times \nabla \boldsymbol{m}) \cdot \nabla \boldsymbol{\varphi} \\ &\quad - \sum_{i=1}^3 \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) \cdot \left(\frac{\partial(\boldsymbol{m} \cdot \boldsymbol{\varphi})}{\partial x_i} \boldsymbol{m} + (\boldsymbol{m} \cdot \boldsymbol{\varphi}) \frac{\partial \boldsymbol{m}}{\partial x_i} \right) \\ &= (\boldsymbol{m} \times \nabla \boldsymbol{m}) \cdot \nabla \boldsymbol{\varphi}, \end{aligned}$$

where in the last step we used the elementary property $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0$ for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$.

Now consider each term on the left-hand side of (4.4). By using (4.12)–(4.15) and noting (2.8) we obtain

$$\begin{aligned}
\langle \mathbf{m}_t, \boldsymbol{\xi} \rangle_{D_T} &= \lambda_1 \langle \mathbf{m}_t, \boldsymbol{\varphi} \rangle_{D_T} + \lambda_2 \langle \mathbf{m}_t, \boldsymbol{\varphi} \times \mathbf{m} \rangle_{D_T} \\
&= -\lambda_1 \langle \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_t), \boldsymbol{\varphi} \rangle_{D_T} - \lambda_2 \langle \mathbf{m}_t, \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T} \\
&= \lambda_1 \langle \mathbf{m} \times \mathbf{m}_t, \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T} - \lambda_2 \langle \mathbf{m}_t, \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T}, \\
\lambda_1 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla \boldsymbol{\xi} \rangle_{D_T} &= \lambda_1^2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla \boldsymbol{\varphi} \rangle_{D_T} + \lambda_1 \lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\boldsymbol{\varphi} \times \mathbf{m}) \rangle_{D_T} \\
&= -\lambda_1^2 \langle \nabla \mathbf{m}, \mathbf{m} \times \nabla \boldsymbol{\varphi} \rangle_{D_T} + \lambda_1 \lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\boldsymbol{\varphi} \times \mathbf{m}) \rangle_{D_T}, \\
\lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\mathbf{m} \times \boldsymbol{\xi}) \rangle_{D_T} &= \lambda_1 \lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\mathbf{m} \times \boldsymbol{\varphi}) \rangle_{D_T} \\
&\quad + \lambda_2^2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\mathbf{m} \times (\boldsymbol{\varphi} \times \mathbf{m})) \rangle_{D_T} \\
&= -\lambda_1 \lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\boldsymbol{\varphi} \times \mathbf{m}) \rangle_{D_T} + \lambda_2^2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla \boldsymbol{\varphi} \rangle_{D_T} \\
&= -\lambda_1 \lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla (\boldsymbol{\varphi} \times \mathbf{m}) \rangle_{D_T} - \lambda_2^2 \langle \nabla \mathbf{m}, \mathbf{m} \times \nabla \boldsymbol{\varphi} \rangle_{D_T}, \\
- \langle F(t, \mathbf{m}), \boldsymbol{\xi} \rangle_{D_T} &= -\lambda_1 \langle F(t, \mathbf{m}), \boldsymbol{\varphi} \rangle_{D_T} - \lambda_2 \langle F(t, \mathbf{m}), \boldsymbol{\varphi} \times \mathbf{m} \rangle_{D_T} \\
&= -\lambda_1^2 \left\langle \mathbf{m} \times \tilde{C}(W(t), \mathbf{m}), \boldsymbol{\varphi} \right\rangle_{D_T} \\
&\quad + \lambda_2^2 \left\langle \mathbf{m} \times (\mathbf{m} \times \tilde{C}(W(t), \mathbf{m})), \boldsymbol{\varphi} \times \mathbf{m} \right\rangle_{D_T} \\
&= - \langle R(t, \mathbf{m}), \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T}, \\
- \lambda_1 \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, \boldsymbol{\xi} \rangle_{D_T} &= \lambda_1^2 \langle e^{-W(t)G} \mathbf{P}, \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T} \\
&\quad - \lambda_1 \lambda_2 \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, \boldsymbol{\varphi} \times \mathbf{m} \rangle_{D_T}, \\
\lambda_2 \langle \mathbf{m} \times (\mathbf{m} \times e^{-W(t)G} \mathbf{P}), \boldsymbol{\xi} \rangle_{D_T} &= \lambda_1 \lambda_2 \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, \boldsymbol{\varphi} \times \mathbf{m} \rangle_{D_T} \\
&\quad - \lambda_2^2 \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, \mathbf{m} \times (\boldsymbol{\varphi} \times \mathbf{m}) \rangle_{D_T} \\
&= \lambda_1 \lambda_2 \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, \boldsymbol{\varphi} \times \mathbf{m} \rangle_{D_T} \\
&\quad - \lambda_2^2 \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, \boldsymbol{\varphi} \rangle_{D_T} \\
&= \lambda_1 \lambda_2 \langle \mathbf{m} \times e^{-W(t)G} \mathbf{P}, \boldsymbol{\varphi} \times \mathbf{m} \rangle_{D_T} \\
&\quad + \lambda_2^2 \langle e^{-W(t)G} \mathbf{P}, \mathbf{m} \times \boldsymbol{\varphi} \rangle_{D_T}.
\end{aligned}$$

Adding the above equations side by side we deduce that the left-hand side of (4.4) equals that of (4.11). Thus (4.4) holds if (4.11) holds. The lemma is proved. \square

Thanks to Lemma 4.1 and Lemma 4.2, in order to solve (1.5)–(1.6), we solve (4.11) and (4.5). It is therefore necessary to define the weak martingale solutions for these two latter equations.

Definition 4.3. *Given $T \in (0, \infty)$, a weak martingale solution to (4.11) and (4.5) on the time interval $[0, T]$, denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}, W, \mathbf{m}, \mathbf{P})$, consists of*

- (a) *a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ with the filtration satisfying the usual conditions,*

- (b) a one-dimensional (\mathcal{F}_t) -adapted Wiener process $W = (W_t)_{t \in [0, T]}$,
- (c) a progressively measurable process $\mathbf{m} : [0, T] \times \Omega \rightarrow \mathbb{L}^2(D)$,
- (d) a progressively measurable process $\mathbf{P} : [0, T] \times \Omega \rightarrow \mathbb{L}^2(\tilde{D})$,

such that there hold

- (1) $\mathbf{m} \in \mathbb{H}^1(D_T)$, \mathbb{P} -a.s.;
- (2) $\mathbf{P} \in L^2(0, T; \mathbb{H}(\text{curl}; \tilde{D}))$, \mathbb{P} -a.s.;
- (3) $\mathbb{E}(\text{ess sup}_{t \in [0, T]} \|\nabla \mathbf{m}(t)\|_D^2) < \infty$;
- (4) $|\mathbf{m}(t, \cdot)| = 1$ for all $t \in [0, T]$, a.e. in D , and \mathbb{P} -a.s.;
- (5) (\mathbf{m}, \mathbf{P}) satisfies (4.11) and (4.5) \mathbb{P} -a.s.

We state the following lemma which is a direct consequence of Lemma 4.1, Lemma 4.2, and statement (4.3).

Lemma 4.4. *If (\mathbf{m}, \mathbf{P}) is a weak martingale solution of (4.11) and (4.5) in the sense of Definition 4.3, then (\mathbf{M}, \mathbf{H}) is a weak martingale solution of (1.5) and (1.6) in the sense of Definition 3.1.*

In the next section, we present a finite element scheme to approximate the solutions of (4.11) and (4.5).

5. THE FINITE ELEMENT SCHEME

In this section we introduce the θ -linear finite element scheme which approximates a weak solution (\mathbf{m}, \mathbf{P}) defined in Definition 4.3.

Let \mathbb{T}_h be a regular tetrahedrization of the domain \tilde{D} into tetrahedra of maximal mesh-size h . Let $\mathbb{T}_h|_D$ be its restriction to $D \subset \tilde{D}$. We denote by $\mathcal{N}_h := \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ the set of vertices in $\mathbb{T}_h|_D$ and by $\mathcal{M}_h := \{\mathbf{e}_1, \dots, \mathbf{e}_M\}$ the set of edges in \mathbb{T}_h .

To discretize the equation (4.11), we introduce the finite element space $\mathbb{V}_h \subset \mathbb{H}^1(D)$ defined by

$$\mathbb{V}_h := \left\{ \mathbf{u} \in \mathbb{H}^1(D) : \mathbf{u}|_K \in (P_1|_K)^3 \quad \forall K \in \mathbb{T}_K \right\},$$

where P_1 is the set of polynomials of maximum total degree 1 in x_1, x_2, x_3 . A basis for \mathbb{V}_h can be chosen to be $\{\phi_n \boldsymbol{\xi}_1, \phi_n \boldsymbol{\xi}_2, \phi_n \boldsymbol{\xi}_3\}_{1 \leq n \leq N}$, where ϕ_n is a continuous piecewise linear function on \mathbb{T}_h satisfying $\phi_n(\mathbf{x}_m) = \delta_{n,m}$ (the Kronecker delta) and $\{\boldsymbol{\xi}_j\}_{j=1, \dots, 3}$ is the canonical basis for \mathbb{R}^3 . The interpolation operator from $\mathbb{C}^0(D)$ onto \mathbb{V}_h is defined by

$$I_{\mathbb{V}_h}(\mathbf{v}) = \sum_{n=1}^N \mathbf{v}(\mathbf{x}_n) \phi_n(\mathbf{x}) \quad \forall \mathbf{v} \in \mathbb{C}^0(D, \mathbb{R}^3).$$

To discretize (4.5), we introduce the lowest order edge elements of Nédélec's first family (see [24]) defined by

$$\mathbb{Y}_h := \left\{ \mathbf{u} \in \mathbb{H}(\text{curl}; \tilde{D}) : \mathbf{u}|_K \in \mathcal{D}_K \quad \forall K \in \mathbb{T}_h \right\},$$

where

$$\mathcal{D}_K := \left\{ \mathbf{v} : K \rightarrow \mathbb{R}^3 : \exists \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \text{ such that } \mathbf{v}(\mathbf{x}) = \mathbf{a} + \mathbf{b} \times \mathbf{x} \quad \forall \mathbf{x} \in K \right\}.$$

A basis $\{\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_M\}$ of \mathbb{Y}_h can be defined by

$$\int_{e_p} \boldsymbol{\psi}_q \cdot \boldsymbol{\tau}_p ds = \delta_{q,p},$$

where $\boldsymbol{\tau}_p$ is the unit vector in the direction of edge e_p . For any $\delta > 0$ and $p > 2$, the interpolation operator $I_{\mathbb{Y}_h}$ from $\mathbb{H}^{1/2+\delta}(\tilde{D}) \cap \mathbb{W}^{1,p}(\tilde{D})$ onto \mathbb{Y}_h is defined by

$$I_{\mathbb{Y}_h}(\mathbf{u}) = \sum_{q=1}^M u_q \boldsymbol{\psi}_q \quad \forall \mathbf{u} \in \mathbb{H}^{1/2+\delta}(\tilde{D}) \cap \mathbb{W}^{1,p}(\tilde{D}),$$

where $u_q = \int_{e_q} \mathbf{u} \cdot \boldsymbol{\tau}_q ds$.

Before introducing our approximation scheme, we state the following result, proved in [5], which will be used in the analysis.

Lemma 5.1. *If there holds*

$$(5.1) \quad \int_D \nabla \phi_i \cdot \nabla \phi_j d\mathbf{x} \leq 0 \quad \text{for all } i, j \in \{1, 2, \dots, N\} \text{ and } i \neq j,$$

then for all $\mathbf{u} \in \mathbb{V}_h$ satisfying $|\mathbf{u}(\mathbf{x}_l)| \geq 1$, $l = 1, 2, \dots, N$, there holds

$$(5.2) \quad \int_D \left| \nabla I_{\mathbb{V}_h} \left(\frac{\mathbf{u}}{|\mathbf{u}|} \right) \right|^2 d\mathbf{x} \leq \int_D |\nabla \mathbf{u}|^2 d\mathbf{x}.$$

When $d = 2$, condition (5.1) holds for Delaunay triangulations. When $d = 3$, it holds if all dihedral angles of the tetrahedra in $\mathbb{T}_h|_D$ are less than or equal to $\pi/2$; see [5]. In the sequel we assume that (5.1) holds.

With the finite element spaces defined as above, we are ready to define our approximation scheme. Fixing a positive integer J , we choose the time step k to be $k = T/J$ and define $t_j = jk$, $j = 0, \dots, J$. For $j = 1, 2, \dots, J$, the functions $\mathbf{m}(t_j, \cdot)$ and $\mathbf{P}(t_j, \cdot)$ are approximated by $\mathbf{m}_h^{(j)} \in \mathbb{V}_h$ and $\mathbf{P}_h^{(j)} \in \mathbb{Y}_h$, respectively. If $\mathbf{v}_h^{(j)}$ is an approximation of $\mathbf{m}_t(t_j, \cdot)$, then since

$$\mathbf{m}_t(t_j, \cdot) \approx \frac{\mathbf{m}(t_{j+1}, \cdot) - \mathbf{m}(t_j, \cdot)}{k} \approx \frac{\mathbf{m}_h^{(j+1)} - \mathbf{m}_h^{(j)}}{k},$$

we can define $\mathbf{m}_h^{(j+1)}$ from $\mathbf{m}_h^{(j)}$ by

$$(5.3) \quad \mathbf{m}_h^{(j+1)} := \mathbf{m}_h^{(j)} + k\mathbf{v}_h^{(j)},$$

To maintain the condition $|\mathbf{m}_h^{(j+1)}| = 1$, we normalise the right-hand side of (5.3) and therefore define $\mathbf{m}_h^{(j+1)}$ belonging to \mathbb{V}_h by

$$\mathbf{m}_h^{(j+1)} = I_{\mathbb{V}_h} \left(\frac{\mathbf{m}_h^{(j)} + k\mathbf{v}_h^{(j)}}{|\mathbf{m}_h^{(j)} + k\mathbf{v}_h^{(j)}|} \right) = \sum_{n=1}^N \frac{\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j)}(\mathbf{x}_n)}{|\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j)}(\mathbf{x}_n)|} \phi_n,$$

which ensures that $|\mathbf{m}_h^{(j+1)}| = 1$ at vertices. Hence it suffices to propose a scheme to compute $\mathbf{v}_h^{(j)}$.

We first rewrite (4.11) as

$$(5.4) \quad \begin{aligned} & \lambda_2 \langle \mathbf{m}_t, \mathbf{w} \rangle_{D_T} - \lambda_1 \langle \mathbf{m} \times \mathbf{m}_t, \mathbf{w} \rangle_{D_T} + \mu \langle \nabla \mathbf{m}, \nabla \mathbf{w} \rangle_{D_T} \\ & = - \langle R(t, \mathbf{m}), \mathbf{w} \rangle_{D_T} + \mu \langle e^{-W(t)G} \mathbf{P}, \mathbf{w} \rangle_{D_T} \end{aligned}$$

where $\mathbf{w} = \mathbf{m} \times \boldsymbol{\varphi}$. Then, noting that $\mathbf{m}_t \cdot \mathbf{m} = 0$ (which follows from $|\mathbf{m}| = 1$) and $\mathbf{w} \cdot \mathbf{m} = 0$, we can design a Galerkin method in which the unknown $\mathbf{v}_h^{(j)}$ and the test function \mathbf{w}_h reflect the above property. Hence we follow [1, 3] to define

$$\mathbb{W}_h^{(j)} := \left\{ \mathbf{w} \in \mathbb{V}_h \mid \mathbf{w}(\mathbf{x}_n) \cdot \mathbf{m}_h^{(j)}(\mathbf{x}_n) = 0, \ n = 1, \dots, N \right\},$$

and we will seek $\mathbf{v}_h^{(j)}$ in this space. It remains to approximate the other terms in (5.4).

Considering the piecewise constant approximation $W_k(t)$ of $W(t)$, namely,

$$(5.5) \quad W_k(t) = W(t_j), \quad t \in [t_j, t_{j+1}),$$

we define

$$(5.6) \quad \begin{aligned} \mathbf{g}_h &:= I_{\mathbb{V}_h}(\mathbf{g}), \\ G_h \mathbf{u} &:= \mathbf{u} \times \mathbf{g}_h \quad \forall \mathbf{u} \in \mathbb{V}_h \cup \mathbb{Y}_h, \\ e^{W_k(t)G_h} \mathbf{u} &:= \mathbf{u} + (\sin W_k(t))G_h \mathbf{u} + (1 - \cos W_k(t))G_h^2 \mathbf{u} \quad \forall \mathbf{u} \in \mathbb{V}_h \cup \mathbb{Y}_h, \\ C_h(\mathbf{u}) &:= \mathbf{u} \times I_{\mathbb{V}_h}(\Delta \mathbf{g}) + 2\nabla \mathbf{u} \times I_{\mathbb{V}_h}(\nabla \mathbf{g}) \quad \forall \mathbf{u} \in \mathbb{V}_h, \end{aligned}$$

$$(5.7) \quad D_{h,k}(t, \mathbf{u}) = \left((\sin W_k(t))C_h + (1 - \cos W_k(t))(G_h C_h + C_h G_h) \right) \mathbf{u}$$

$$(5.8) \quad \tilde{C}_{h,k}(t, \mathbf{u}) = \left(I - \sin W_k(t)G_h + (1 - \cos W_k(t))G_h^2 \right) D_{h,k}(t, \mathbf{u}),$$

$$(5.9) \quad R_{h,k}(t, \mathbf{u}) = \lambda_2^2 \mathbf{u} \times (\mathbf{u} \times \tilde{C}_{h,k}(t, \mathbf{u})) - \lambda_1^2 \tilde{C}_{h,k}(t, \mathbf{u}).$$

We can now discretise (5.4) as: For some $\theta \in [0, 1]$, find $\mathbf{v}_h^{(j)} \in \mathbb{W}_h^{(j)}$ satisfying

$$(5.10) \quad \begin{aligned} & \lambda_2 \langle \mathbf{v}_h^{(j)}, \mathbf{w}_h^{(j)} \rangle_D - \lambda_1 \langle \mathbf{m}_h^{(j)} \times \mathbf{v}_h^{(j)}, \mathbf{w}_h^{(j)} \rangle_D + \mu \langle \nabla(\mathbf{m}_h^{(j)} + k\theta \mathbf{v}_h^{(j)}), \nabla \mathbf{w}_h^{(j)} \rangle_D \\ & = - \langle R_{h,k}(t_j, \mathbf{m}_h^{(j)}), \mathbf{w}_h \rangle_D + \mu \langle e^{-W_k(t_j)G_h} \mathbf{P}_h^{(j)}, \mathbf{w}_h^{(j)} \rangle_D \quad \forall \mathbf{w}_h^{(j)} \in \mathbb{W}_h^{(j)}. \end{aligned}$$

To discretise (4.5), even though \mathbf{P} is not time differentiable we formally use integration by parts to bring the time derivative to \mathbf{P} , and thus with $d_t \mathbf{P}_h^{(j+1)}$ defined by

$$d_t \mathbf{P}_h^{(j+1)} := k^{-1}(\mathbf{P}_h^{(j+1)} - \mathbf{P}_h^{(j)}),$$

the discretisation of (4.5) reads: Compute $\mathbf{P}_h^{(j+1)} \in \mathbb{Y}_h$ by solving

$$(5.11) \quad \mu_0 \langle d_t \mathbf{P}_h^{(j+1)}, \boldsymbol{\zeta}_h \rangle_{\tilde{D}} + \langle \sigma \nabla \times \mathbf{P}_h^{(j+1)}, \nabla \times \boldsymbol{\zeta}_h \rangle_{\tilde{D}} = \sigma_D \langle \nabla \times (e^{W_k(t_j)G_h} \mathbf{m}_h^{(j)}), \nabla \times \boldsymbol{\zeta}_h \rangle_D \quad \forall \boldsymbol{\zeta}_h \in \mathbb{Y}_h.$$

We summarise the above procedure in the following algorithm.

Algorithm 5.1.

Step 1: Set $j = 0$. Choose $\mathbf{m}_h^{(0)} = I_{\mathbb{V}_h} \mathbf{m}_0$ and $\mathbf{P}_h^{(0)} = I_{\mathbb{Y}_h} \mathbf{P}_0$.

Step 2: Solve (5.10) and (5.11) to find $(\mathbf{v}_h^{(j)}, \mathbf{P}_h^{(j+1)}) \in \mathbb{W}_h^{(j)} \times \mathbb{Y}_h$.

Step 3: Define

$$\mathbf{m}_h^{(j+1)}(\mathbf{x}) := \sum_{n=1}^N \frac{\mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j)}(\mathbf{x}_n)}{\left| \mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j)}(\mathbf{x}_n) \right|} \phi_n(\mathbf{x}).$$

Step 4: Set $j = j + 1$ and return to Step 2 if $j < J$. Stop if $j = J$.

By the Lax–Milgram theorem, for each $j > 0$ there exists a unique solution $(\mathbf{v}_h^{(j)}, \mathbf{P}_h^{(j+1)}) \in \mathbb{W}_h^{(j)} \times \mathbb{Y}_h$ of equations (5.10)–(5.11). Since $\left| \mathbf{m}_h^{(0)}(\mathbf{x}_n) \right| = 1$ and $\mathbf{v}_h^{(j)}(\mathbf{x}_n) \cdot \mathbf{m}_h^{(j)}(\mathbf{x}_n) = 0$ for all $n = 1, \dots, N$ and $j = 0, \dots, J$, there hold (by induction)

$$(5.12) \quad \left| \mathbf{m}_h^{(j)}(\mathbf{x}_n) + k\mathbf{v}_h^{(j+1)}(\mathbf{x}_n) \right| \geq 1 \quad \text{and} \quad \left| \mathbf{m}_h^{(j)}(\mathbf{x}_n) \right| = 1, \quad j = 0, \dots, J.$$

In particular, the above inequality shows that Step 3 of the algorithm is well defined.

We finish this section by proving the following lemmas concerning boundedness of $\mathbf{m}_h^{(j)}$, $\mathbf{P}_h^{(j)}$ and $R_{h,k}$.

Lemma 5.2. For any $j = 0, \dots, J$ there hold

$$\|\mathbf{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \leq 1 \quad \text{and} \quad \|\mathbf{m}_h^{(j)}\|_D \leq |D|,$$

where $|D|$ denotes the measure of D .

Proof. The first inequality follows from (5.12) and the second can be obtained by integrating over D . \square

Lemma 5.3. Assume that \mathbf{g} satisfies (1.7) and $\mathbf{g} \in \mathbb{W}^{2,\infty}(D)$. There exists a deterministic constant c depending only on \mathbf{g} such that, for any $j = 0, \dots, J$, there holds \mathbb{P} -a.s.,

$$(5.13) \quad \left\| R_{h,k}(t_j, \mathbf{m}_h^{(j)}) \right\|_D^2 \leq c + c \left\| \nabla \mathbf{m}_h^{(j)} \right\|_D^2,$$

$$(5.14) \quad \|e^{-W_k(t_j)G_h} \mathbf{u}\|_D^2 \leq \|\mathbf{u}\|_D^2 \quad \forall \mathbf{u} \in \mathbb{L}^2(D),$$

$$(5.15) \quad \|\nabla \times (e^{W_k(t_j)G_h} \mathbf{m}_h^{(j)})\|_D^2 \leq c + c \|\nabla \mathbf{m}_h^{(j)}\|_D^2.$$

Proof. The proof of (5.13) is similar to that of [14, Lemma 5.3]. To prove (5.14) we first note that the definition of $e^{-W_k(t_j)G_h} \mathbf{u}$ gives

$$\begin{aligned} |e^{-W_k(t_j)G_h} \mathbf{u}|^2 &= \left| \mathbf{u} - (\sin W_k(t_j)) \mathbf{u} \times \mathbf{g}_h + (1 - \cos W_k(t_j)) (\mathbf{u} \times \mathbf{g}_h) \times \mathbf{g}_h \right|^2 \\ &= |\mathbf{u}|^2 + (1 - \cos W_k(t_j))^2 (|\mathbf{u} \times \mathbf{g}_h|^2 - |\mathbf{u} \times \mathbf{g}_h|^2) \\ &= |\mathbf{u}|^2 + (1 - \cos W_k(t_j))^2 (|\mathbf{g}_h|^2 - 1) |\mathbf{u} \times \mathbf{g}_h|^2, \end{aligned}$$

where in the last step we used $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}|^2 = |\mathbf{a} \times \mathbf{b}|^2 |\mathbf{b}|^2$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Since $|\mathbf{g}(\mathbf{x}_i)| = 1$ and $\sum_{i=1}^N \phi_i(\mathbf{x}) = 1$ for all $\mathbf{x} \in D$, we have

$$|\mathbf{g}_h(\mathbf{x})|^2 = \left| \sum_{i=1}^N \mathbf{g}(\mathbf{x}_i) \phi_i(\mathbf{x}) \right|^2 \leq 1.$$

Therefore,

$$\left| e^{-W_k(t_j)G_h} \mathbf{u} \right|^2 \leq |\mathbf{u}|^2 \quad a.e. \text{ in } D,$$

proving (5.14).

Finally, in order to prove (5.15) we use the inequality

$$\|\nabla \times \mathbf{u}\|_D^2 \leq c \|\nabla \mathbf{u}\|_D^2 \quad \forall \mathbf{u} \in \mathbb{H}^1(D)$$

to obtain

$$\|\nabla \times (e^{W_k(t_j)G_h} \mathbf{m}_h^{(j)})\|_D^2 \leq c \|\nabla (e^{W_k(t_j)G_h} \mathbf{m}_h^{(j)})\|_D^2.$$

On the other hand from the definition of $e^{W_k(t_j)G_h}$, we deduce

$$\begin{aligned} \nabla (e^{W_k(t_j)G_h} \mathbf{m}_h^{(j)}) &= e^{W_k(t_j)G_h} \nabla \mathbf{m}_h^{(j)} + (\sin W_k(t_j)) \mathbf{m}_h^{(j)} \times \nabla \mathbf{g}_h \\ &\quad + (1 - \cos W_k(t_j)) \left((\mathbf{m}_h^{(j)} \times \nabla \mathbf{g}_h) \times \mathbf{g}_h + (\mathbf{m}_h^{(j)} \times \mathbf{g}_h) \times \nabla \mathbf{g}_h \right). \end{aligned}$$

Since $\mathbf{g} \in \mathbb{W}^{2,\infty}(D)$, by using Lemma 5.2 and (5.14) we obtain from the above equality

$$\left| \nabla (e^{W_k(t_j)G_h} \mathbf{m}_h^{(j)}) \right|^2 \leq c + \left| e^{W_k(t_j)G_h} (\nabla \mathbf{m}_h^{(j)}) \right|^2 \leq c + c \left| \nabla \mathbf{m}_h^{(j)} \right|^2.$$

This completes the proof. \square

Lemma 5.4. *The sequence $\left\{ (\mathbf{m}_h^{(j)}, \mathbf{v}_h^{(j)}, \mathbf{P}_h^{(j)}) \right\}_{j=0,1,\dots,J}$ produced by Algorithm 5.1 satisfies \mathbb{P} -a.s.,*

$$\begin{aligned} (5.16) \quad & \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + k \sum_{i=0}^{j-1} \|\mathbf{v}_h^{(i)}\|_D^2 + k^2(2\theta - 1) \sum_{i=0}^{j-1} \|\nabla \mathbf{v}_h^{(i)}\|_D^2 + \|\mathbf{P}_h^{(j)}\|_{\tilde{D}}^2 \\ & + \sum_{i=0}^{j-1} \|\mathbf{P}_h^{(i+1)} - \mathbf{P}_h^{(i)}\|_{\tilde{D}}^2 + k \sum_{i=0}^{j-1} \|\nabla \times \mathbf{P}_h^{(i)}\|_{\tilde{D}}^2 \leq c. \end{aligned}$$

Proof. Choosing $\mathbf{w}_h^{(j)} = \mathbf{v}_h^{(j)}$ in (5.10), we obtain

$$\begin{aligned} \lambda_2 \|\mathbf{v}_h^{(j)}\|_D^2 + \mu k \theta \|\nabla \mathbf{v}_h^{(j)}\|_D^2 &= -\mu \left\langle \nabla \mathbf{m}_h^{(j)}, \nabla \mathbf{v}_h^{(j)} \right\rangle_D - \left\langle R_{h,k}(t_j, \mathbf{m}_h^{(j)}), \mathbf{v}_h^{(j)} \right\rangle_D \\ &\quad + \mu \left\langle e^{-W_k(t_j)G_h} \mathbf{P}_h^{(j)}, \mathbf{v}_h^{(j)} \right\rangle_D, \end{aligned}$$

or equivalently

$$\begin{aligned} \left\langle \nabla \mathbf{m}_h^{(j)}, \nabla \mathbf{v}_h^{(j)} \right\rangle_D &= -\lambda_2 \mu^{-1} \|\mathbf{v}_h^{(j)}\|_D^2 - k \theta \|\nabla \mathbf{v}_h^{(j)}\|_D^2 - \mu^{-1} \left\langle R_{h,k}(t_j, \mathbf{m}_h^{(j)}), \mathbf{v}_h^{(j)} \right\rangle_D \\ &\quad + \left\langle e^{-W_k(t_j)G_h} \mathbf{P}_h^{(j)}, \mathbf{v}_h^{(j)} \right\rangle_D. \end{aligned}$$

Lemma 5.1 and the above equation yield

$$\begin{aligned} \|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 &\leq \|\nabla(\mathbf{m}_h^{(j)} + k\mathbf{v}_h^{(j)})\|_D^2 \\ &= \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + k^2(1 - 2\theta)\|\nabla \mathbf{v}_h^{(j)}\|_D^2 - 2k\lambda_2\mu^{-1}\|\mathbf{v}_h^{(j)}\|_D^2 \\ &\quad - 2k\mu^{-1}\left\langle R_{h,k}(t_j, \mathbf{m}_h^{(j)}), \mathbf{v}_h^{(j)} \right\rangle_D + 2k\left\langle e^{-W_k(t_j)G_h} \mathbf{P}_h^{(j)}, \mathbf{v}_h^{(j)} \right\rangle_D. \end{aligned}$$

By using the elementary inequality

$$(5.17) \quad 2ab \leq \alpha^{-1}a^2 + \alpha b^2 \quad \forall \alpha > 0, \forall a, b \in \mathbb{R},$$

for the last two terms on the right hand side, we deduce

$$\begin{aligned} &\|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 + 2k\lambda_2\mu^{-1}\|\mathbf{v}_h^{(j)}\|_D^2 + k^2(2\theta - 1)\|\nabla \mathbf{v}_h^{(j)}\|_D^2 \\ &\leq \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + k\lambda_2\mu^{-1}\|\mathbf{v}_h^{(j)}\|_D^2 + 2k\lambda_2^{-1}\mu^{-1}\|R_{h,k}(t_j, \mathbf{m}_h^{(j)})\|_D^2 + 2k\lambda_2^{-1}\mu\|e^{-W_k(t_j)G_h} \mathbf{P}_h^{(j)}\|_D^2. \end{aligned}$$

By rearranging the above inequality and using (5.13)–(5.14) we obtain

$$\begin{aligned} &\|\nabla \mathbf{m}_h^{(j+1)}\|_D^2 + k\lambda_2\mu^{-1}\|\mathbf{v}_h^{(j)}\|_D^2 + k^2(2\theta - 1)\|\nabla \mathbf{v}_h^{(j)}\|_D^2 \\ &\leq \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + 2k\lambda_2^{-1}\mu\|\mathbf{P}_h^{(j)}\|_D^2 + k\lambda_2^{-1}\mu^{-1}c\|\nabla \mathbf{m}_h^{(j)}\|_D^2 + k\lambda_2^{-1}\mu^{-1}c. \end{aligned}$$

Replacing j by i in the above inequality and summing for i from 0 to $j - 1$ yields

$$\begin{aligned} &\|\nabla \mathbf{m}_h^{(j)}\|_D^2 + \lambda_2\mu^{-1}k \sum_{i=0}^{j-1} \|\mathbf{v}_h^{(i+1)}\|_D^2 + (2\theta - 1)k^2 \sum_{i=0}^{j-1} \|\nabla \mathbf{v}_h^{(i+1)}\|_D^2 \\ &\leq \|\nabla \mathbf{m}_h^{(0)}\|_D^2 + ck \sum_{i=0}^{j-1} \|\mathbf{P}_h^{(i)}\|_D^2 + ck \sum_{i=0}^{j-1} \|\nabla \mathbf{m}_h^{(i)}\|_D^2 + c. \end{aligned}$$

Since $\mathbf{m}_0 \in \mathbb{H}^2(D)$ it can be shown that there exists a deterministic constant c depending only on \mathbf{m}_0 such that

$$(5.18) \quad \|\nabla \mathbf{m}_h^{(0)}\|_D^2 \leq c.$$

By using (5.18) we deduce

$$\begin{aligned} &\|\nabla \mathbf{m}_h^{(j)}\|_D^2 + k \sum_{i=0}^{j-1} \|\mathbf{v}_h^{(i+1)}\|_D^2 + k^2(2\theta - 1) \sum_{i=0}^{j-1} \|\nabla \mathbf{v}_h^{(i+1)}\|_D^2 \\ (5.19) \quad &\leq c + c \sum_{i=0}^{j-1} k\|\mathbf{P}_h^{(i)}\|_D^2 + c \sum_{i=0}^{j-1} k\|\nabla \mathbf{m}_h^{(i)}\|_D^2. \end{aligned}$$

In order to estimate the two sums on the right-hand side, we take $\boldsymbol{\zeta}_h = \mathbf{P}_h^{(j+1)}$ in (5.11) to obtain the following identity

$$\mu_0 \left\langle d_t \mathbf{P}_h^{(j+1)}, \mathbf{P}_h^{(j+1)} \right\rangle_{\bar{D}} + \left\langle \sigma \nabla \times \mathbf{P}_h^{(j+1)}, \nabla \times \mathbf{P}_h^{(j+1)} \right\rangle_{\bar{D}} = \sigma_D \left\langle \nabla \times (e^{W_k(t_j)G_h} \mathbf{m}_h^{(j)}), \nabla \times \mathbf{P}_h^{(j+1)} \right\rangle_D.$$

Let σ_0 is the lower bound of σ on \tilde{D} . By using successively (5.17) and (5.15) we deduce from the above equality

$$\begin{aligned} \mu_0 \left\langle \mathbf{P}_h^{(j+1)} - \mathbf{P}_h^{(j)}, \mathbf{P}_h^{(j+1)} \right\rangle_{\tilde{D}} + k\sigma_0 \|\nabla \times \mathbf{P}_h^{(j+1)}\|_{\tilde{D}}^2 &\leq +k \frac{\sigma_D^2}{2\sigma_0} \|\nabla \times (e^{W_k(t_j)G_h} \mathbf{m}_h^{(j)})\|_D^2 \\ &\quad + \frac{1}{2} k\sigma_0 \|\nabla \times \mathbf{P}_h^{(j+1)}\|_{\tilde{D}}^2 \\ &\leq \frac{1}{2} k\sigma_0 \|\nabla \times \mathbf{P}_h^{(j+1)}\|_{\tilde{D}}^2 \\ &\quad + ck \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + ck, \end{aligned}$$

or equivalently

$$\mu_0 \left\langle \mathbf{P}_h^{(j+1)} - \mathbf{P}_h^{(j)}, \mathbf{P}_h^{(j+1)} \right\rangle_{\tilde{D}} + \frac{1}{2} k\sigma_0 \|\nabla \times \mathbf{P}_h^{(j+1)}\|_{\tilde{D}}^2 \leq ck \|\nabla \mathbf{m}_h^{(j)}\|_D^2 + ck.$$

Replacing j by i in the above inequality and summing over i from 0 to $j-1$ and using the following Abel summation

$$\sum_{i=0}^{j-1} (\mathbf{a}_{i+1} - \mathbf{a}_i) \cdot \mathbf{a}_{i+1} = \frac{1}{2} |\mathbf{a}_j|^2 - \frac{1}{2} |\mathbf{a}_0|^2 + \frac{1}{2} \sum_{i=0}^{j-1} |\mathbf{a}_{i+1} - \mathbf{a}_i|^2, \quad \mathbf{a}_i \in \mathbb{R}^3,$$

we obtain

$$\begin{aligned} \|\mathbf{P}_h^{(j)}\|_{\tilde{D}}^2 + \sum_{i=0}^{j-1} \|\mathbf{P}_h^{(i+1)} - \mathbf{P}_h^{(i)}\|_{\tilde{D}}^2 + \sigma_0 \mu_0^{-1} \sum_{i=0}^{j-1} k \|\nabla \times \mathbf{P}_h^{(i+1)}\|_{\tilde{D}}^2 \\ \leq \|\mathbf{P}_h^{(0)}\|_{\tilde{D}}^2 + c \sum_{i=0}^{j-1} k \|\nabla \mathbf{m}_h^{(i)}\|_D^2 + cT\sigma. \end{aligned}$$

By using (3.3) and the error estimate for the interpolant $\mathbf{P}_h^{(0)} = I_{\mathbb{Y}_h} \mathbf{P}_0$, it can be shown that there exists a constant c depending only on \mathbf{P}_0 such that

$$(5.20) \quad \|\mathbf{P}_h^{(0)}\|_{\tilde{D}}^2 + \|\nabla \times \mathbf{P}_h^{(0)}\|_{\tilde{D}}^2 \leq c.$$

By using (5.20) we deduce

$$(5.21) \quad \|\mathbf{P}_h^{(j)}\|_{\tilde{D}}^2 + \sum_{i=0}^{j-1} \|\mathbf{P}_h^{(i+1)} - \mathbf{P}_h^{(i)}\|_{\tilde{D}}^2 + k \sum_{i=0}^{j-1} \|\nabla \times \mathbf{P}_h^{(i+1)}\|_{\tilde{D}}^2 \leq c + ck \sum_{i=0}^{j-1} \|\nabla \mathbf{m}_h^{(i)}\|_D^2.$$

From (5.19) and (5.21) we obtain

$$\|\nabla \mathbf{m}_h^{(j)}\|_D^2 + \|\mathbf{P}_h^{(j)}\|_{\tilde{D}}^2 \leq c + ck \sum_{i=0}^{j-1} \|\mathbf{P}_h^{(i)}\|_{\tilde{D}}^2 + ck \sum_{i=0}^{j-1} \|\nabla \mathbf{m}_h^{(i)}\|_D^2.$$

By using induction and (5.18)-(5.20) we can show that

$$\|\nabla \mathbf{m}_h^{(i)}\|_D^2 + \|\mathbf{P}_h^{(i)}\|_{\tilde{D}}^2 \leq c(1 + ck)^i.$$

Summing over i from 0 to $j-1$ and using $1+x \leq e^x$ we obtain

$$(5.22) \quad k \sum_{i=0}^{j-1} \|\nabla \mathbf{m}_h^{(i)}\|_D^2 + k \sum_{i=0}^{j-1} \|\mathbf{P}_h^{(i)}\|_{\tilde{D}}^2 \leq ck \frac{(1+ck)^j - 1}{ck} \leq e^{ckJ} = c.$$

The required result (5.16) now follows from (5.19), (5.21) and (5.22). \square

6. PROOF OF THE MAIN THEOREM

The discrete solutions $\mathbf{m}_h^{(j)}$, $\mathbf{v}_h^{(j)}$ and $\mathbf{P}_h^{(j)}$ constructed via Algorithm 5.1 are interpolated in time in the following definition.

Definition 6.1. *For all $x \in D$ and all $t \in [0, T]$, let $j \in \{0, \dots, J-1\}$ be such that $t \in [t_j, t_{j+1})$. We then define*

$$\begin{aligned} \mathbf{m}_{h,k}(t, \mathbf{x}) &:= \frac{t - t_j}{k} \mathbf{m}_h^{(j+1)}(\mathbf{x}) + \frac{t_{j+1} - t}{k} \mathbf{m}_h^{(j)}(\mathbf{x}), \\ \mathbf{m}_{h,k}^-(t, \mathbf{x}) &:= \mathbf{m}_h^{(j)}(\mathbf{x}), \\ \mathbf{v}_{h,k}(t, \mathbf{x}) &:= \mathbf{v}_h^{(j)}(\mathbf{x}), \\ \mathbf{P}_{h,k}(t, \mathbf{x}) &:= \frac{t - t_j}{k} \mathbf{P}_h^{(j+1)}(\mathbf{x}) + \frac{t_{j+1} - t}{k} \mathbf{P}_h^{(j)}(\mathbf{x}), \\ \mathbf{P}_{h,k}^-(t, \mathbf{x}) &:= \mathbf{P}_h^{(j)}(\mathbf{x}), \\ \mathbf{P}_{h,k}^+(t, \mathbf{x}) &:= \mathbf{P}_h^{(j+1)}(\mathbf{x}). \end{aligned}$$

The above sequences have the following obvious bounds.

Lemma 6.2. *There exist a deterministic constant c depending on \mathbf{m}_0 , \mathbf{P}_0 , \mathbf{g} , μ , σ and T such that for all $\theta \in [0, 1]$ there holds \mathbb{P} -a.s.*

$$\|\mathbf{m}_{h,k}^*\|_{D_T}^2 + \|\nabla \mathbf{m}_{h,k}^*\|_{D_T}^2 + \|\mathbf{v}_{h,k}\|_{D_T}^2 + k(2\theta - 1) \|\nabla \mathbf{v}_{h,k}\|_{D_T}^2 \leq c,$$

where $\mathbf{m}_{h,k}^* = \mathbf{m}_{h,k}$ or $\mathbf{m}_{h,k}^-$. In particular, when $\theta \in [0, \frac{1}{2})$, there holds \mathbb{P} -a.s.

$$\|\mathbf{m}_{h,k}^*\|_{D_T}^2 + \|\nabla \mathbf{m}_{h,k}^*\|_{D_T}^2 + (1 + (2\theta - 1)kh^{-2}) \|\mathbf{v}_{h,k}\|_{D_T}^2 \leq c.$$

Proof. Both inequalities are direct consequences of Definition 6.1, Lemmas 5.2 and 5.4, noting that the second inequality requires the use of the inverse estimate (see e.g. [19])

$$\|\nabla \mathbf{v}_h^{(i)}\|_D^2 \leq ch^{-2} \|\mathbf{v}_h^{(i)}\|_D^2.$$

\square

Lemma 6.3. *There exist a deterministic constant c depending on \mathbf{m}_0 , \mathbf{P}_0 , \mathbf{g} , μ , σ and T such that for all $\theta \in [0, 1]$ there holds \mathbb{P} -a.s.*

$$(6.1) \quad \|\mathbf{P}_{h,k}\|_{\tilde{D}_T}^2 + \|\mathbf{P}_{h,k}^+\|_{\tilde{D}_T}^2 + \|\nabla \times \mathbf{P}_{h,k}^+\|_{\tilde{D}_T}^2 \leq c,$$

$$(6.2) \quad \|\mathbf{P}_{h,k} - \mathbf{P}_{h,k}^*\|_{\tilde{D}_T}^2 \leq kc,$$

where $\mathbf{P}_{h,k}^* = \mathbf{P}_{h,k}^+$ or $\mathbf{P}_{h,k}^-$.

Proof. It is easy to prove (6.1) by using Lemma 5.4 and Definition 6.1. Inequality (6.2) can be deduced from Lemma 5.4 by noting that for $t \in [t_j, t_{j+1})$ there holds

$$|\mathbf{P}_{h,k}(t, \mathbf{x}) - \mathbf{P}_{h,k}^+(t, \mathbf{x})| = \left| \frac{t - t_{j+1}}{k} (\mathbf{P}_h^{(j+1)}(\mathbf{x}) - \mathbf{P}_h^{(j)}(\mathbf{x})) \right| \leq |\mathbf{P}_h^{(j+1)}(\mathbf{x}) - \mathbf{P}_h^{(j)}(\mathbf{x})|,$$

completing the proof of the lemma. \square

The next lemma provides a bound of $\mathbf{m}_{h,k}$ in the \mathbb{H}^1 -norm and relationships between $\mathbf{m}_{h,k}^-$, $\mathbf{m}_{h,k}$ and $\mathbf{v}_{h,k}$.

Lemma 6.4. *Assume that h and k go to 0 with a further condition $k = o(h^2)$ when $\theta \in [0, \frac{1}{2})$ and no condition otherwise. The sequences $\{\mathbf{m}_{h,k}\}$, $\{\mathbf{m}_{h,k}^-\}$, and $\{\mathbf{v}_{h,k}\}$ defined in Definition 6.1 satisfy the following properties \mathbb{P} -a.s.*

$$(6.3) \quad \|\mathbf{m}_{h,k}\|_{\mathbb{H}^1(D_T)} \leq c,$$

$$(6.4) \quad \|\mathbf{m}_{h,k} - \mathbf{m}_{h,k}^-\|_{D_T} \leq ck,$$

$$(6.5) \quad \|\mathbf{v}_{h,k} - \partial_t \mathbf{m}_{h,k}\|_{\mathbb{L}^1(D_T)} \leq ck,$$

$$(6.6) \quad \|\mathbf{m}_{h,k} - 1\|_{D_T} \leq chk.$$

Proof. The proof of this lemma is similar to that of [14, Lemma 6.3] \square

The following two Lemmas 6.5 show that $\mathbf{m}_{h,k}$ and $\mathbf{P}_{h,k}$, respectively, satisfy discrete forms of (4.11) and (4.5).

Lemma 6.5. *Assume that h and k go to 0 with the following conditions*

$$(6.7) \quad \begin{cases} k = o(h^2) & \text{when } 0 \leq \theta < 1/2, \\ k = o(h) & \text{when } \theta = 1/2, \\ \text{no condition} & \text{when } 1/2 < \theta \leq 1. \end{cases}$$

Then for any $\varphi \in C(0, T; \mathbb{C}^\infty(D))$ and $\zeta \in C_T^1(0, T; \mathbb{C}_\times^\infty(\tilde{D}))$, there holds \mathbb{P} -a.s.

$$(6.8) \quad \begin{aligned} & -\lambda_1 \langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, \mathbf{m}_{h,k}^- \times \varphi \rangle_{D_T} + \lambda_2 \langle \mathbf{v}_{h,k}, \mathbf{m}_{h,k}^- \times \varphi \rangle_{D_T} \\ & + \mu \langle \nabla(\mathbf{m}_{h,k}^- + k\theta \mathbf{v}_{h,k}), \nabla(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T} + \langle R_{h,k}(\cdot, \mathbf{m}_{h,k}^-), \mathbf{m}_{h,k}^- \times \varphi \rangle_{D_T} \\ & - \mu \langle e^{W_k G_h} \mathbf{P}_{h,k}^-, \mathbf{m}_{h,k}^- \times \varphi \rangle_{D_T} = O(h + k) \end{aligned}$$

and

$$(6.9) \quad \begin{aligned} & \mu_0 \langle \mathbf{P}_{h,k}, \zeta_t \rangle_{\tilde{D}_T} - \mu_0 \left\langle \mathbf{P}_h^{(0)}, \zeta(0, \cdot) \right\rangle_{\tilde{D}} - \langle \sigma \nabla \times \mathbf{P}_{h,k}^+, \nabla \times \zeta \rangle_{\tilde{D}_T} \\ & + \sigma_D \langle e^{W_k G_h} \mathbf{m}_{h,k}^-, \nabla \times (\nabla \times \zeta) \rangle_{D_T} = O(h + k). \end{aligned}$$

Proof.

Proof of (6.8): For $t \in [t_j, t_{j+1})$, we use (5.10) with $\mathbf{w}_h^{(j)} = I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^-(t, \cdot) \times \varphi(t, \cdot)) \in \mathbb{W}_h^{(j)}$ to have

$$\begin{aligned} & -\lambda_1 \langle \mathbf{m}_{h,k}^-(t, \cdot) \times \mathbf{v}_{h,k}(t, \cdot), I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^-(t, \cdot) \times \varphi(t, \cdot)) \rangle_D \\ & + \lambda_2 \langle \mathbf{v}_{h,k}(t, \cdot), I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^-(t, \cdot) \times \varphi(t, \cdot)) \rangle_D \\ & + \mu \langle \nabla(\mathbf{m}_{h,k}^-(t, \cdot) + k\theta \mathbf{v}_{h,k}(t, \cdot)), \nabla I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^-(t, \cdot) \times \varphi(t, \cdot)) \rangle_D \\ & + \langle R_{h,k}(t_j, \mathbf{m}_{h,k}^-(t, \cdot)), I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^-(t, \cdot) \times \varphi(t, \cdot)) \rangle_D \\ & - \mu \langle e^{W_k(t)G_h} \mathbf{P}_{h,k}^-(t), I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^-(t, \cdot) \times \varphi(t, \cdot)) \rangle_D = 0. \end{aligned}$$

Integrating both sides of the above equation over (t_j, t_{j+1}) and summing over $j = 0, \dots, J-1$ we deduce

$$\begin{aligned} & -\lambda_1 \langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T} + \lambda_2 \langle \mathbf{v}_{h,k}, I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T} \\ & + \mu \langle \nabla(\mathbf{m}_{h,k}^- + k\theta \mathbf{v}_{h,k}), \nabla I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T} + \langle R_{h,k}(\cdot, \mathbf{m}_{h,k}^-), I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T} \\ & - \mu \langle e^{W_k G_h} \mathbf{P}_{h,k}^-, I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T} = 0. \end{aligned}$$

This implies

$$\begin{aligned} & -\lambda_1 \langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, \mathbf{m}_{h,k}^- \times \varphi \rangle_{D_T} + \lambda_2 \langle \mathbf{v}_{h,k}, \mathbf{m}_{h,k}^- \times \varphi \rangle_{D_T} \\ & + \mu \langle \nabla(\mathbf{m}_{h,k}^- + k\theta \mathbf{v}_{h,k}), \nabla(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T} + \langle R_{h,k}(\cdot, \mathbf{m}_{h,k}^-), \mathbf{m}_{h,k}^- \times \varphi \rangle_{D_T} \\ & - \mu \langle e^{W_k G_h} \mathbf{P}_{h,k}^-, \mathbf{m}_{h,k}^- \times \varphi \rangle_{D_T} = I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \langle -\lambda_1 \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k} + \lambda_2 \mathbf{v}_{h,k}, \mathbf{m}_{h,k}^- \times \varphi - I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T}, \\ I_2 &= \mu \langle \nabla(\mathbf{m}_{h,k}^- + k\theta \mathbf{v}_{h,k}), \nabla(\mathbf{m}_{h,k}^- \times \varphi - I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi)) \rangle_{D_T}, \\ I_3 &= \langle R_{h,k}(\cdot, \mathbf{m}_{h,k}^-), \mathbf{m}_{h,k}^- \times \varphi - I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T}, \\ I_4 &= -\langle e^{W_k G_h} \mathbf{P}_{h,k}^-, \mathbf{m}_{h,k}^- \times \varphi - I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi) \rangle_{D_T}. \end{aligned}$$

Hence it suffices to prove that $I_i = O(h+k)$ for $i = 1, \dots, 4$. Firstly, by using Lemma 5.2 we obtain

$$\|\mathbf{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \leq \sup_{0 \leq j \leq J} \|\mathbf{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \leq 1.$$

This inequality, Lemma 6.2 and Lemma 8.2 yield

$$\begin{aligned} |I_1| &\leq c (\|\mathbf{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} + 1) \|\mathbf{v}_{h,k}\|_{D_T} \|\mathbf{m}_{h,k}^- \times \varphi - I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi)\|_{D_T} \\ &\leq c \|\mathbf{m}_{h,k}^- \times \varphi - I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \varphi)\|_{D_T} \leq ch. \end{aligned}$$

The bounds for I_2 , I_3 and I_4 can be obtained similarly by using Lemma 6.2 and Lemma 5.3, respectively, noting that when $\theta \in [0, \frac{1}{2}]$, a bound of $\|\nabla \mathbf{v}_{h,k}\|_{D_T}$ can be deduced from the inverse estimate $\|\nabla \mathbf{v}_{h,k}\|_{D_T} \leq ch^{-1} \|\mathbf{v}_{h,k}\|_{D_T}$. This completes the proof (6.8).

Proof of (6.9): For $t \in [t_j, t_{j+1})$, we use (5.11) with $\zeta_h(t, \cdot) = I_{\mathbb{Y}_h} \zeta(t, \cdot)$ to have

$$\begin{aligned} \mu_0 \langle \partial_t \mathbf{P}_{h,k}(t, \cdot), I_{\mathbb{Y}_h} \zeta(t, \cdot) \rangle_{\tilde{D}} &= - \langle \sigma \nabla \times \mathbf{P}_{h,k}^+(t, \cdot), \nabla \times I_{\mathbb{Y}_h} \zeta(t, \cdot) \rangle_{\tilde{D}} \\ &\quad + \sigma_D \langle \nabla \times e^{W_k(t)G_h} \mathbf{m}_{h,k}^-(t, \cdot), \nabla \times I_{\mathbb{Y}_h} \zeta(t, \cdot) \rangle_D. \end{aligned}$$

Integrating both sides of the above equation over (t_j, t_{j+1}) and summing over $j = 0, \dots, J-1$, and using integration by parts (noting that $\zeta_h(T, \cdot) = 0$) we deduce

$$\begin{aligned} \mu_0 \langle \mathbf{P}_{h,k}, \partial_t \zeta_h \rangle_{\tilde{D}_T} - \mu_0 \langle \mathbf{P}_h^{(0)}, \zeta_h(0, \cdot) \rangle_{\tilde{D}} &= \langle \sigma \nabla \times \mathbf{P}_{h,k}^+, \nabla \times \zeta_h \rangle_{\tilde{D}_T} \\ &\quad - \sigma_D \langle \nabla \times e^{W_k G_h} \mathbf{m}_{h,k}^-, \nabla \times \zeta_h \rangle_{D_T}. \end{aligned}$$

By using Lemma 6.3 and the following error estimate, see e.g. [24],

$$\|\zeta(t) - \zeta_h(t)\|_{\tilde{D}} + h \|\nabla \times (\zeta(t) - \zeta_h(t))\|_{\tilde{D}} \leq Ch^2 \|\nabla^2 \zeta\|_{\tilde{D}},$$

we deduce

$$\begin{aligned} \mu_0 \langle \mathbf{P}_{h,k}, \zeta_t \rangle_{\tilde{D}_T} - \mu_0 \langle \mathbf{P}_h^{(0)}, \zeta(0, \cdot) \rangle_{\tilde{D}} &- \langle \sigma \nabla \times \mathbf{P}_{h,k}^+, \nabla \times \zeta \rangle_{\tilde{D}_T} \\ &+ \sigma_D \langle \nabla \times e^{W_k G_h} \mathbf{m}_{h,k}^-, \nabla \times \zeta \rangle_{D_T} = O(h). \end{aligned}$$

Using Green's identity (see [24, Corollary 3.20]) we obtain (6.9), completing the proof of the lemma. \square

In the next lemma we show that $\mathbf{v}_{h,k}$ can be replaced by $\partial_t \mathbf{m}_{h,k}$, as indeed the latter approximates \mathbf{m}_t .

Lemma 6.6. *Assume that h and k go to 0 satisfying (6.7). Then for any $\varphi \in C_0^1(0, T; C^\infty(D))$ and $\zeta \in C_T^1(0, T; \mathbb{C}_\infty^\infty(\tilde{D}))$, there holds \mathbb{P} -a.s.*

$$\begin{aligned} &- \lambda_1 \langle \mathbf{m}_{h,k} \times \partial_t \mathbf{m}_{h,k}, \mathbf{m}_{h,k} \times \varphi \rangle_{D_T} + \lambda_2 \langle \partial_t \mathbf{m}_{h,k}, \mathbf{m}_{h,k} \times \varphi \rangle_{D_T} \\ &+ \mu \langle \nabla \mathbf{m}_{h,k}, \nabla (\mathbf{m}_{h,k} \times \varphi) \rangle_{D_T} + \langle R_{h,k}(\cdot, \mathbf{m}_{h,k}), \mathbf{m}_{h,k} \times \varphi \rangle_{D_T} \\ (6.10) \quad &- \mu \langle e^{W_k G_h} \mathbf{P}_{h,k}^+, \mathbf{m}_{h,k} \times \varphi \rangle_{D_T} = O(h+k); \end{aligned}$$

and

$$\begin{aligned} \mu_0 \langle \mathbf{P}_{h,k}^+, \zeta_t \rangle_{\tilde{D}_T} - \mu_0 \langle \mathbf{P}_h^{(0)}, \zeta(0, \cdot) \rangle_{\tilde{D}} &- \langle \sigma \nabla \times \mathbf{P}_{h,k}^+, \nabla \times \zeta \rangle_{\tilde{D}_T} \\ (6.11) \quad &+ \sigma_D \langle e^{W_k G_h} \mathbf{m}_{h,k}, \nabla \times (\nabla \times \zeta) \rangle_{D_T} = O(h+k). \end{aligned}$$

Proof.

Proof of (6.10): From (6.8) it follows that

$$\begin{aligned} &- \lambda_1 \langle \mathbf{m}_{h,k} \times \partial_t \mathbf{m}_{h,k}, \mathbf{m}_{h,k} \times \varphi \rangle_{D_T} + \lambda_2 \langle \partial_t \mathbf{m}_{h,k}, \mathbf{m}_{h,k} \times \varphi \rangle_{D_T} \\ &+ \mu \langle \nabla (\mathbf{m}_{h,k}), \nabla (\mathbf{m}_{h,k} \times \varphi) \rangle_{D_T} + \langle R_{h,k}(\cdot, \mathbf{m}_{h,k}), \mathbf{m}_{h,k} \times \varphi \rangle_{D_T} \\ &- \mu \langle e^{W_k G_h} \mathbf{P}_{h,k}^+, \mathbf{m}_{h,k} \times \varphi \rangle_{D_T} = O(h+k) + I_1 + \dots + I_5, \end{aligned}$$

where

$$\begin{aligned}
I_1 &= \lambda_1 \langle \mathbf{m}_{h,k}^- \times \mathbf{v}_{h,k}, \mathbf{m}_{h,k}^- \times \boldsymbol{\varphi} \rangle_{D_T} - \lambda_1 \langle \mathbf{m}_{h,k} \times \partial_t \mathbf{m}_{h,k}, \mathbf{m}_{h,k} \times \boldsymbol{\varphi} \rangle_{D_T}, \\
I_2 &= -\lambda_2 \langle \mathbf{v}_{h,k}, \mathbf{m}_{h,k}^- \times \boldsymbol{\varphi} \rangle_{D_T} + \lambda_2 \langle \partial_t \mathbf{m}_{h,k}, \mathbf{m}_{h,k} \times \boldsymbol{\varphi} \rangle_{D_T}, \\
I_3 &= -\mu \langle \nabla(\mathbf{m}_{h,k}^- + k\theta \mathbf{v}_{h,k}), \nabla(\mathbf{m}_{h,k}^- \times \boldsymbol{\varphi}) \rangle_{D_T} + \mu \langle \nabla(\mathbf{m}_{h,k}), \nabla(\mathbf{m}_{h,k} \times \boldsymbol{\varphi}) \rangle_{D_T}, \\
I_4 &= -\langle R_{h,k}(\cdot, \mathbf{m}_{h,k}^-), \mathbf{m}_{h,k}^- \times \boldsymbol{\varphi} \rangle_{D_T} + \langle R_{h,k}(\cdot, \mathbf{m}_{h,k}), \mathbf{m}_{h,k} \times \boldsymbol{\varphi} \rangle_{D_T}, \\
I_5 &= -\mu \langle e^{W_k G_h} \mathbf{P}_{h,k}^+, \mathbf{m}_{h,k} \times \boldsymbol{\varphi} \rangle_{D_T} + \mu \langle e^{W_k G_h} \mathbf{P}_{h,k}^-, \mathbf{m}_{h,k}^- \times \boldsymbol{\varphi} \rangle_{D_T}.
\end{aligned}$$

Hence it suffices to prove that $I_i = O(k)$ for $i = 1, \dots, 5$.

First, by using the triangle inequality we obtain

$$\begin{aligned}
\lambda_1^{-1} |I_1| &\leq \left| \langle (\mathbf{m}_{h,k}^- - \mathbf{m}_{h,k}) \times \mathbf{v}_{h,k}, \mathbf{m}_{h,k}^- \times \boldsymbol{\varphi} \rangle_{D_T} \right| + \left| \langle \mathbf{m}_{h,k} \times \mathbf{v}_{h,k}, (\mathbf{m}_{h,k}^- - \mathbf{m}_{h,k}) \times \boldsymbol{\varphi} \rangle_{D_T} \right| \\
&\quad + \left| \langle \mathbf{m}_{h,k} \times (\mathbf{v}_{h,k} - \partial_t \mathbf{m}_{h,k}), \mathbf{m}_{h,k} \times \boldsymbol{\varphi} \rangle_{D_T} \right| \\
&\leq 2 \|\mathbf{m}_{h,k}^- - \mathbf{m}_{h,k}\|_{D_T} \|\mathbf{v}_{h,k}\|_{D_T} \|\mathbf{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\varphi}\|_{\mathbb{L}^\infty(D_T)} \\
&\quad + \|\mathbf{v}_{h,k} - \partial_t \mathbf{m}_{h,k}\|_{\mathbb{L}^1(D_T)} \|\mathbf{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\varphi}\|_{\mathbb{L}^\infty(D_T)}.
\end{aligned}$$

Therefore, the bound of I_1 can be obtained by using Lemmas 6.2 and 6.4. The bounds for I_2, I_3 and I_4 can be obtained similarly.

Finally, using (5.14), Lemmas 6.2 and 6.3 we obtain

$$\begin{aligned}
\mu^{-1} |I_5| &\leq \left| \langle e^{W_k G_h} (\mathbf{P}_{h,k}^+ - \mathbf{P}_{h,k}^-), \mathbf{m}_{h,k} \times \boldsymbol{\varphi} \rangle_{D_T} \right| + \left| \langle e^{W_k G_h} \mathbf{P}_{h,k}^-, (\mathbf{m}_{h,k} - \mathbf{m}_{h,k}^-) \times \boldsymbol{\varphi} \rangle_{D_T} \right| \\
&\leq \|e^{W_k G_h} (\mathbf{P}_{h,k}^+ - \mathbf{P}_{h,k}^-)\|_{D_T} \|\mathbf{m}_{h,k}\|_{D_T} \|\boldsymbol{\varphi}\|_{\mathbb{L}^\infty(D_T)} \\
&\quad + \|e^{W_k G_h} \mathbf{P}_{h,k}^-\|_{D_T} \|\mathbf{m}_{h,k} - \mathbf{m}_{h,k}^-\|_{D_T} \|\boldsymbol{\varphi}\|_{\mathbb{L}^\infty(D_T)} \\
&\leq c \|\mathbf{P}_{h,k}^+ - \mathbf{P}_{h,k}^-\|_{D_T} + c \|\mathbf{P}_{h,k}^-\|_{D_T} \|\mathbf{m}_{h,k} - \mathbf{m}_{h,k}^-\|_{D_T} \leq ck.
\end{aligned}$$

This completes the proof of (6.10).

Proof of (6.11): It follows from (6.9) that

$$\begin{aligned}
\mu_0 \langle \mathbf{P}_{h,k}^+, \boldsymbol{\zeta}_t \rangle_{\tilde{D}_T} - \mu_0 \left\langle \mathbf{P}_h^{(0)}, \boldsymbol{\zeta}(0, \cdot) \right\rangle_{\tilde{D}} - \langle \sigma \nabla \times \mathbf{P}_{h,k}^+, \nabla \times \boldsymbol{\zeta} \rangle_{\tilde{D}_T} \\
+ \sigma_D \langle e^{W_k G_h} \mathbf{m}_{h,k}, \nabla \times (\nabla \times \boldsymbol{\zeta}) \rangle_{D_T} = O(h+k) + I_6 + I_7,
\end{aligned}$$

where

$$\begin{aligned}
I_6 &= \mu_0 \langle \mathbf{P}_{h,k}^+ - \mathbf{P}_{h,k}, \boldsymbol{\zeta}_t \rangle_{\tilde{D}_T}, \\
I_7 &= \sigma \langle e^{W_k G_h} (\mathbf{m}_{h,k}^- - \mathbf{m}_{h,k}), \nabla \times (\nabla \times \boldsymbol{\zeta}) \rangle_{D_T}.
\end{aligned}$$

By using (6.4) and (6.2) we obtain that $I_i = O(k)$ for $i = 6, 7$. This completes the proof of (6.11). \square

In order to prove the \mathbb{P} -a.s. convergence of random variables $\mathbf{m}_{h,k}$ and $\mathbf{P}_{h,k}^+$, we first show that the family $\mathcal{L}(\mathbf{m}_{h,k})$ and $\mathcal{L}(\mathbf{P}_{h,k}^+)$ are tight.

Lemma 6.7. *Assume that h and k go to 0 satisfying (6.7). Then the set of laws $\{\mathcal{L}(\mathbf{m}_{h,k}, \mathbf{P}_{h,k}^+, W_k)\}$ on the space $C(0, T; \mathbb{H}^{-1}(D)) \times H^{-1}(\tilde{D}_T) \times \mathbb{D}(0, T)$ is tight. Here, $\mathbb{D}(0, T)$ is the Skorokhod space; see e.g. [8].*

Proof. Firstly, from Definition 5.5, the approximation W_k of the Wiener process W belongs to $\mathbb{D}(0, T)$. The tightness of $\{\mathcal{L}(W_k)\}$ in $\mathbb{D}(0, T)$ is proved in [8, Theorem 2.5.6]. The tightness of $\{\mathcal{L}(\mathbf{m}_{h,k})\}$ on $C(0, T; \mathbb{H}^{-1}(D))$ and of $\{\mathcal{L}(\mathbf{P}_{h,k}^+)\}$ on $H^{-1}(\tilde{D}_T)$ can be obtained as in the proof of [14, Lemma 6.6] and is therefore omitted. \square

The following proposition is a consequence of the tightness of $\{\mathcal{L}(\mathbf{m}_{h,k})\}$, $\{\mathcal{L}(\mathbf{P}_{h,k}^+)\}$ and $\{\mathcal{L}(W_k)\}$.

Proposition 6.8. *Assume that h and k go to 0 satisfying (6.7). Then there exist*

- (a) *a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$,*
- (b) *a sequence $\{(\mathbf{m}'_{h,k}, \mathbf{P}'_{h,k}, W'_k)\}$ of random variables defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ and taking values in the space $C(0, T; \mathbb{H}^{-1}(D)) \times \mathbb{H}^{-1}(\tilde{D}_T) \times \mathbb{D}(0, T)$,*
- (c) *a random variable $(\mathbf{m}', \mathbf{P}', W')$ defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ and taking values in $C([0, T]; \mathbb{H}^{-1}(D)) \times \mathbb{H}^{-1}(\tilde{D}_T) \times \mathbb{D}(0, T)$,*

satisfying

- (1) $\mathcal{L}(\mathbf{m}_{h,k}, \mathbf{P}_{h,k}^+, W_k) = \mathcal{L}(\mathbf{m}'_{h,k}, \mathbf{P}'_{h,k}, W'_k)$,
- (2) $\mathbf{m}'_{h,k} \rightarrow \mathbf{m}'$ in $C(0, T; \mathbb{H}^{-1}(D))$ strongly, \mathbb{P}' -a.s.,
- (3) $\mathbf{P}'_{h,k} \rightarrow \mathbf{P}'$ in $\mathbb{H}^{-1}(\tilde{D}_T)$ strongly, \mathbb{P}' -a.s.,
- (4) $W'_k \rightarrow W'$ in $\mathbb{D}(0, T)$ \mathbb{P}' -a.s.

Moreover, the sequence $\{\mathbf{m}'_{h,k}\}$ and $\{\mathbf{P}'_{h,k}\}$ satisfy \mathbb{P}' -a.s.

$$(6.12) \quad \|\mathbf{m}'_{h,k}(\omega')\|_{\mathbb{H}^1(D_T)} \leq c,$$

$$(6.13) \quad \|\mathbf{m}'_{h,k}(\omega')\|_{\mathbb{L}^\infty(D_T)} \leq c,$$

$$(6.14) \quad \|\|\mathbf{m}'_{h,k}(\omega')\| - 1\|_{\mathbb{L}^2(D_T)} \leq c(h + k),$$

$$(6.15) \quad \text{and } \|\mathbf{P}'_{h,k}(\omega)\|_{L^2(0, T; \mathbb{H}(\text{curl}; \tilde{D}))} \leq c.$$

Proof. By Lemma 6.7 and the Donsker theorem [8, Theorem 8.2], the family of probability measures $\{\mathcal{L}(\mathbf{m}_{h,k}, \mathbf{P}_{h,k}^+, W_k)\}$ is tight on $C(0, T; \mathbb{H}^{-1}(D)) \times \mathbb{H}^{-1}(\tilde{D}_T) \times \mathbb{D}(0, T)$. Then by Theorem 5.1 in [8] the family of measures $\{\mathcal{L}(\mathbf{m}_{h,k}, \mathbf{P}_{h,k}^+, W_k)\}$ is relatively compact on $C(0, T; \mathbb{H}^{-1}(D)) \times \mathbb{H}^{-1}(\tilde{D}_T) \times \mathbb{D}(0, T)$, that is there exists a subsequence, still denoted by $\{\mathcal{L}(\mathbf{m}_{h,k}, \mathbf{P}_{h,k}^+, W_k)\}$, such that $\{\mathcal{L}(\mathbf{m}_{h,k}, \mathbf{P}_{h,k}^+, W_k)\}$ converges weakly. Hence, the existence of (a)–(c) satisfying (1)–(4) follows immediately from the Skorokhod Theorem [8, Theorem 6.7] since $C(0, T; \mathbb{H}^{-1}(D)) \times \mathbb{H}^{-1}(\tilde{D}_T) \times \mathbb{D}(0, T)$ is a separable metric space.

We note that from the Kuratowski theorem, the Borel subsets of $\mathbb{H}^1(D_T)$ or $\mathbb{H}^1(D_T) \cap \mathbb{L}^\infty(D_T)$ are Borel subsets of $C(0, T; \mathbb{H}^{-1}(D))$ and the Borel subsets of $L^2(0, T; \mathbb{H}(\text{curl}; \tilde{D}))$ are Borel subsets of $\mathbb{H}^{-1}(\tilde{D}_T)$. The estimates (6.12)–(6.15) are direct consequences of Lemmas 6.3–6.4 and the equality of laws stated in part (1). \square

We now ready to prove the main result of this paper.

Theorem 6.9. *Assume that $T > 0$, $\mathbf{M}_0 \in \mathbb{H}^2(D)$ and $\mathbf{g} \in \mathbb{W}^{2,\infty}(D)$ satisfy (??) and (1.7), respectively. Then \mathbf{m}' , \mathbf{P}' , W' , the sequences $\{\mathbf{m}'_{h,k}\}$, $\{\mathbf{P}'_{h,k}\}$ and the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ given by Proposition 6.8 satisfy*

- (1) *the sequence $\{\mathbf{m}'_{h,k}\}$ converges to \mathbf{m}' weakly in $\mathbb{H}^1(D_T)$, \mathbb{P}' -a.s.*
- (2) *the sequence $\{\mathbf{P}'_{h,k}\}$ converges to \mathbf{P}' weakly in $L^2(0, T; \mathbb{H}(\text{curl}; \tilde{D}))$, \mathbb{P}' -a.s.*
- (3) *$(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \in [0, T]}, \mathbb{P}', W', \mathbf{M}', \mathbf{P}')$ is a weak martingale solution of (1.5), where*

$$\mathbf{M}'(t) := e^{W'(t)G} \mathbf{m}'(t) \quad \forall t \in [0, T], \text{ a.e. } \mathbf{x} \in D.$$

Proof. By Proposition 6.8 there exists a set $V \subset \Omega'$ such that $\mathbb{P}'(V) = 1$,

$$\mathbf{m}'_{h,k}(\omega') \rightarrow \mathbf{m}'(\omega') \quad \text{strongly in } C(0, T; \mathbb{H}^{-1}(D)),$$

$$\mathbf{P}'_{h,k}(\omega') \rightarrow \mathbf{P}'(\omega') \quad \text{strongly in } \mathbb{H}^{-1}(\tilde{D}_T),$$

and (6.12), (6.15) hold for every $\omega' \in V$. In what follows, we work with a fixed $\omega' \in V$.

The convergences of sequences $\{\mathbf{m}'_{h,k}(\omega')\}$ and $\{\mathbf{P}'_{h,k}(\omega')\}$ are obtained by using the same arguments as in [15, Theorem 6.8].

In order to prove (3), by noting Lemma 4.4 we need to prove that \mathbf{m}' , \mathbf{P}' and W' satisfy (4.10), (4.11) and (4.5).

Prove that \mathbf{m}' satisfies (4.10): Since $\mathbb{H}^1(D_T)$ is compactly embedded in $\mathbb{L}^2(D_T)$, there exists a subsequence of $\{\mathbf{m}'_{h,k}(\omega')\}$ (still denoted by $\{\mathbf{m}'_{h,k}(\omega')\}$) such that

$$(6.16) \quad \mathbf{m}'_{h,k}(\omega') \rightarrow \mathbf{m}'(\omega') \quad \text{strongly in } \mathbb{L}^2(D_T).$$

Therefore (4.10) follows from (6.16) and (6.14).

Prove that \mathbf{m}' , \mathbf{P}' satisfy (4.11) and (4.5): From Lemma 6.6, $(\mathbf{m}_{h,k}, \mathbf{P}_{h,k}^+, W_k)$ satisfies (6.10)–(6.11) \mathbb{P} -a.s.. Therefore, it follows from the equality of laws in Proposition 6.8 that $(\mathbf{m}'_{h,k}, \mathbf{P}'_{h,k}, W'_k)$ satisfies the following equations for all $\psi \in C_0^\infty((0, T); \mathbb{C}^\infty(D))$ and $\zeta \in C_c^\infty([0, T], \mathbb{C}_\times^\infty(\tilde{D}))$, \mathbb{P}' -a.s.

$$(6.17) \quad \begin{aligned} & -\lambda_1 \langle \mathbf{m}'_{h,k}(\omega') \times \partial_t \mathbf{m}'_{h,k}(\omega'), \mathbf{m}'_{h,k}(\omega') \times \psi \rangle_{D_T} + \lambda_2 \langle \partial_t \mathbf{m}'_{h,k}(\omega'), \mathbf{m}'_{h,k}(\omega') \times \psi \rangle_{D_T} \\ & + \mu \langle \nabla(\mathbf{m}'_{h,k}(\omega')), \nabla(\mathbf{m}'_{h,k}(\omega') \times \psi) \rangle_{D_T} + \langle R_{h,k}(\cdot, \mathbf{m}'_{h,k}(\omega')), \mathbf{m}'_{h,k}(\omega') \times \psi \rangle_{D_T} \\ & - \mu \left\langle e^{W'_k G_h} \mathbf{P}'_{h,k}(\omega'), \mathbf{m}'_{h,k}(\omega') \times \psi \right\rangle_{D_T} = O(h + k), \end{aligned}$$

and

$$(6.18) \quad \begin{aligned} & \mu_0 \langle \mathbf{P}'_{h,k}(\omega'), \zeta_t \rangle_{\tilde{D}_T} - \mu_0 \left\langle \mathbf{P}_h^{(0)}, \zeta(0, \cdot) \right\rangle_{\tilde{D}_T} + \langle \sigma \nabla \times \mathbf{P}'_{h,k}(\omega'), \nabla \times \zeta \rangle_{\tilde{D}_T} \\ & - \sigma_D \left\langle e^{W'_k G_h} \mathbf{m}'_{h,k}(\omega'), \nabla \times (\nabla \times \zeta) \right\rangle_{D_T} = O(h + k). \end{aligned}$$

It suffices now to use the same arguments as in [15, Theorem 6.8] to pass the limit in (6.17) and (6.18). Indeed, from [15, Theorem 6.8] there hold

$$\begin{aligned} \langle \mathbf{m}'(\omega') \times \partial_t \mathbf{m}'(\omega'), \mathbf{m}'(\omega') \times \boldsymbol{\psi} \rangle_{D_T} &\rightarrow \langle \mathbf{m}'(\omega') \times \partial_t \mathbf{m}'(\omega'), \mathbf{m}'(\omega') \times \boldsymbol{\psi} \rangle_{D_T} \\ \langle \partial_t \mathbf{m}'_{h,k}(\omega'), \mathbf{m}'_{h,k}(\omega') \times \boldsymbol{\psi} \rangle_{D_T} &\rightarrow \langle \partial_t \mathbf{m}'(\omega'), \mathbf{m}'(\omega') \times \boldsymbol{\psi} \rangle_{D_T} \\ \langle \nabla(\mathbf{m}'_{h,k}(\omega')), \nabla(\mathbf{m}'_{h,k}(\omega') \times \boldsymbol{\psi}) \rangle_{D_T} &\rightarrow \langle \nabla(\mathbf{m}'(\omega')), \nabla(\mathbf{m}'(\omega') \times \boldsymbol{\psi}) \rangle_{D_T} \\ \langle R_{h,k}(\cdot, \mathbf{m}'_{h,k}(\omega')), \mathbf{m}'_{h,k}(\omega') \times \boldsymbol{\psi} \rangle_{D_T} &\rightarrow \langle R(\cdot, \mathbf{m}'(\omega')), \mathbf{m}'(\omega') \times \boldsymbol{\psi} \rangle_{D_T}. \end{aligned}$$

To prove the convergence of the last term in (6.17), we use the triangle inequality, Hölder inequality, (5.14), (5.6) and (2.3) to obtain

$$\begin{aligned} \mathcal{I} &:= \left| \left\langle e^{W'_k G_h} \mathbf{P}'_{h,k}(\omega'), \mathbf{m}'_{h,k}(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} - \left\langle e^{W'G} \mathbf{P}'(\omega'), \mathbf{m}'(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} \right| \\ &\leq \left| \left\langle e^{W'_k G_h} \mathbf{P}'_{h,k}(\omega'), (\mathbf{m}'_{h,k}(\omega') - \mathbf{m}'(\omega')) \times \boldsymbol{\psi} \right\rangle_{D_T} \right| \\ &\quad + \left| \left\langle (e^{W'_k G_h} - e^{W'_k G}) \mathbf{P}'_{h,k}(\omega'), \mathbf{m}'(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} \right| \\ &\quad + \left| \left\langle (e^{W'_k G} - e^{W'G}) \mathbf{P}'_{h,k}(\omega'), \mathbf{m}'(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} \right| \\ &\quad + \left| \left\langle e^{W'G} (\mathbf{P}'_{h,k}(\omega') - \mathbf{P}'(\omega')), \mathbf{m}'(\omega') \times \boldsymbol{\psi} \right\rangle_{D_T} \right| \\ &\leq \|\mathbf{P}'_{h,k}(\omega')\|_{D_T} \|\mathbf{m}'_{h,k}(\omega') - \mathbf{m}'(\omega')\|_{D_T} \|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)} \\ &\quad + c \|I_{\mathbb{V}_h}(\mathbf{g}) - \mathbf{g}\|_D \|\mathbf{P}'_{h,k}(\omega')\|_{D_T} \|\mathbf{m}'(\omega')\|_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)} \\ &\quad + c \|W_k(\omega') - W'(\omega')\|_{\mathbb{L}^\infty([0,T])} \|\mathbf{P}'_{h,k}(\omega')\|_{D_T} \|\mathbf{m}'(\omega')\|_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)} \\ &\quad + \left| \left\langle \mathbf{P}'_{h,k}(\omega') - \mathbf{P}'(\omega'), e^{-W'G} (\mathbf{m}'(\omega') \times \boldsymbol{\psi}) \right\rangle_{D_T} \right| \\ &\leq c \|\mathbf{m}'_{h,k}(\omega') - \mathbf{m}'(\omega')\|_{D_T} + c \|I_{\mathbb{V}_h}(\mathbf{g}) - \mathbf{g}\|_D + c \|W_k(\omega') - W'(\omega')\|_{\mathbb{L}^\infty([0,T])} \\ &\quad + \left| \left\langle \mathbf{P}'_{h,k}(\omega') - \mathbf{P}'(\omega'), e^{-W'G} (\mathbf{m}'(\omega') \times \boldsymbol{\psi}) \right\rangle_{D_T} \right|, \end{aligned}$$

here the last inequality is obtained by using (6.15) and $|\mathbf{m}'(\omega')| = 1$ a.e..

Hence, it follows from (6.16), part (4) in Proposition 6.8 and the weak convergence of $\{\mathbf{P}'_{h,k}(\omega')\}$ in $L^2(0, T; \mathbb{H}(\text{curl}; \tilde{D}))$ that

$$\mathcal{I} \rightarrow 0 \quad \text{as } h, k \rightarrow 0.$$

This implies that \mathbf{m}', \mathbf{P}' satisfy (4.11).

The convergence of (6.18) can be proved in the same manner by noting that $\{\mathbf{P}'_{h,k}(\omega')\}$ converges weakly in $L^2(0, T; \mathbb{H}(\text{curl}; \tilde{D}))$, completing the proof of the theorem. \square

7. NUMERICAL EXPERIMENT

In order to carry out physically relevant experiments (see [16]), the initial fields $\mathbf{M}_0, \mathbf{H}_0$ must satisfy the following conditions

$$\operatorname{div}(\mathbf{H}_0 + \widetilde{\mathbf{M}}_0) = 0 \text{ in } \widetilde{D} \quad \text{and} \quad (\mathbf{H}_0 + \widetilde{\mathbf{M}}_0) \cdot \mathbf{n} = 0 \text{ on } \partial\widetilde{D}.$$

This can be achieved by taking

$$\mathbf{H}_0 = \mathbf{H}_0^* - \chi_D \mathbf{M}_0,$$

where $\operatorname{div} \mathbf{H}_0^* = 0$ in \widetilde{D} . In our experiment, for simplicity, we choose \mathbf{H}_0^* to be a constant. We solve an academic example with $D = \widetilde{D} = (0, 1)^3$ and

$$\begin{aligned} \mathbf{M}_0(\mathbf{x}) &= \begin{cases} (0, 0, -1), & |\mathbf{x}^*| \geq \frac{1}{2}, \\ (2\mathbf{x}^*A, A^2 - |\mathbf{x}^*|^2)/(A^2 + |\mathbf{x}^*|^2), & |\mathbf{x}^*| \leq \frac{1}{2}, \end{cases} \\ \mathbf{H}_0^*(\mathbf{x}) &= (0, 0, H_s), \quad \mathbf{x} \in \widetilde{D}, \end{aligned}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{x}^* = (x_1 - 0.5, x_2 - 0.5, 0)$ and $A = (1 - 2|\mathbf{x}^*|)^4/4$. The constant H_s represents the strength of \mathbf{H}_0 in the x_3 -direction. We carried out the experiments for $H_s = 30$. We set the values for the other parameters in (3.1) and (3.2) as $\lambda_1 = \lambda_2 = \mu_0 = \sigma = 1$.

For each time step k , we generate a discrete Brownian path by:

$$W_k(t_{j+1}) - W_k(t_j) \sim \mathcal{N}(0, k) \quad \text{for all } j = 0, \dots, J-1.$$

An approximation of any expected value is computed as the average of L discrete Brownian paths. In our experiments, we choose $L = 400$.

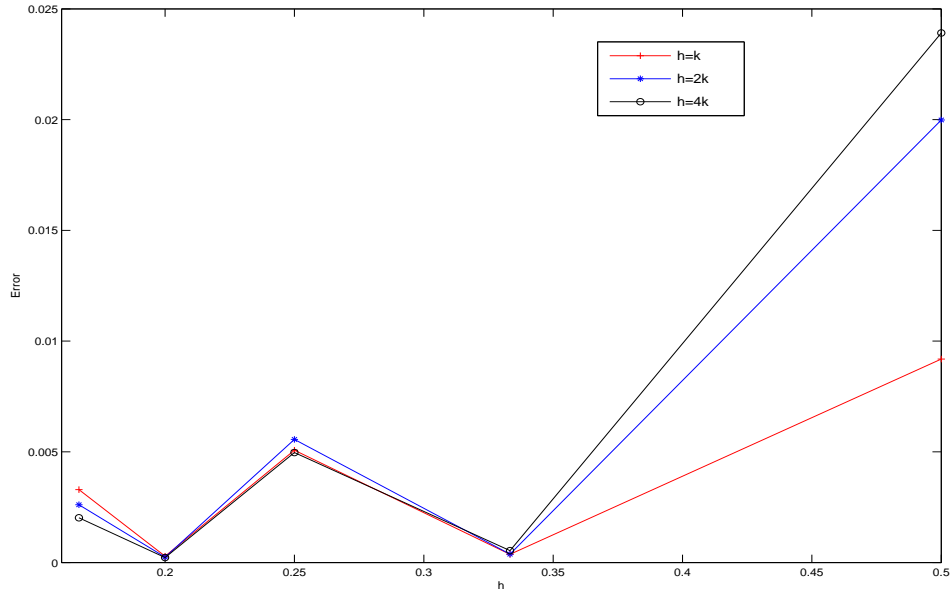
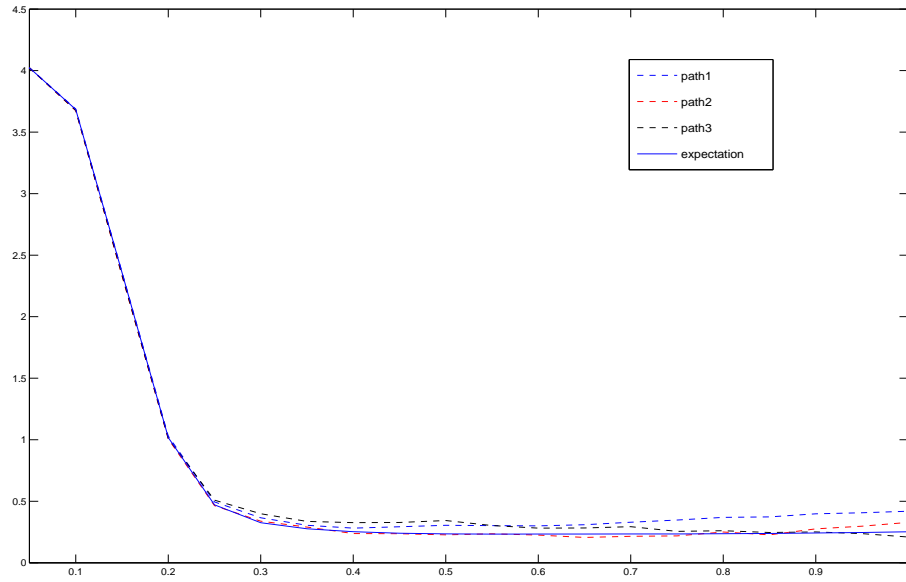
At each iteration we solve two linear systems of sizes $2N \times 2N$ and $M \times M$, recalling that N is the number of vertices and M is the number of edges in the triangulation. The code is written in Fortran90. The parameter θ in Algorithm 5.1 is chosen to be 0.7.

In the first set of experiments, to observe convergence of the method, we solve with $T = 1$, $h = 1/n$ where $n = 2, \dots, 7$, and different time steps $k = h$, $k = h/2$, and $k = h/4$. For each value of h , the domain D is partitioned into uniform cubes of size h . Each cube is then partitioned into six tetrahedra. Noting that

$$E_{h,k}^2 := \int_{D_T} |1 - |\mathbf{m}_{h,k}^-||^2 d\mathbf{x} dt = |||\mathbf{m}| - |\mathbf{m}_{h,k}^-|||_{D_T}^2 \leq \|\mathbf{m} - \mathbf{m}_{h,k}^-\|_{D_T}^2,$$

we compute and plot in Figure 1 the error $\mathbb{E}[E_{h,k}^2]$ for different values of h and k .

In the second set of experiments to observe boundedness of discrete energies, we solve the problem with fixed values of $h = 1/7$ and $k = 1/20$. We plot $t \mapsto \|\nabla \mathbf{m}_{h,k}(t)\|_D^2$ in Figure 2 and $t \mapsto \|\mathbf{P}_{h,k}(t)\|_{\widetilde{D}}^2$ in Figure 3 for three individual paths and the expectations which seems to suggest that these energies are bounded when $t \rightarrow \infty$. Figure 4 shows that the total energy $\mathcal{E}(t) := \|\nabla \mathbf{m}_{h,k}(t)\|_D^2 + \|\mathbf{P}_{h,k}(t)\|_{\widetilde{D}}^2$ is bounded as in Lemma 5.4.

FIGURE 1. Plot of error $\mathbb{E}[E_{h,k}^2]$ FIGURE 2. Plot of $t \mapsto \|\nabla \mathbf{M}_{h,k}(t)\|_D$, expectation and three individual paths

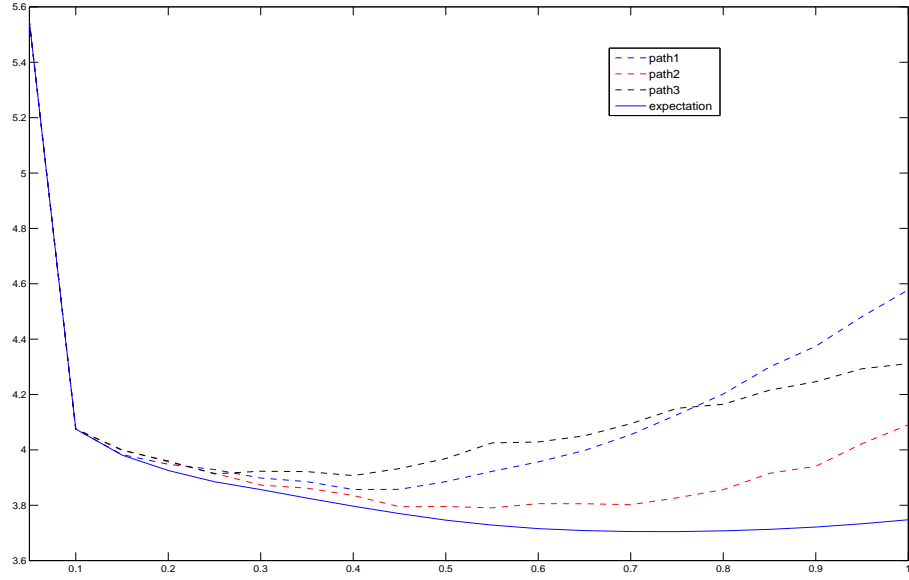


FIGURE 3. Plot of $t \mapsto \|P_{h,k}(t)\|_{\tilde{D}}$, expectation and three individual paths

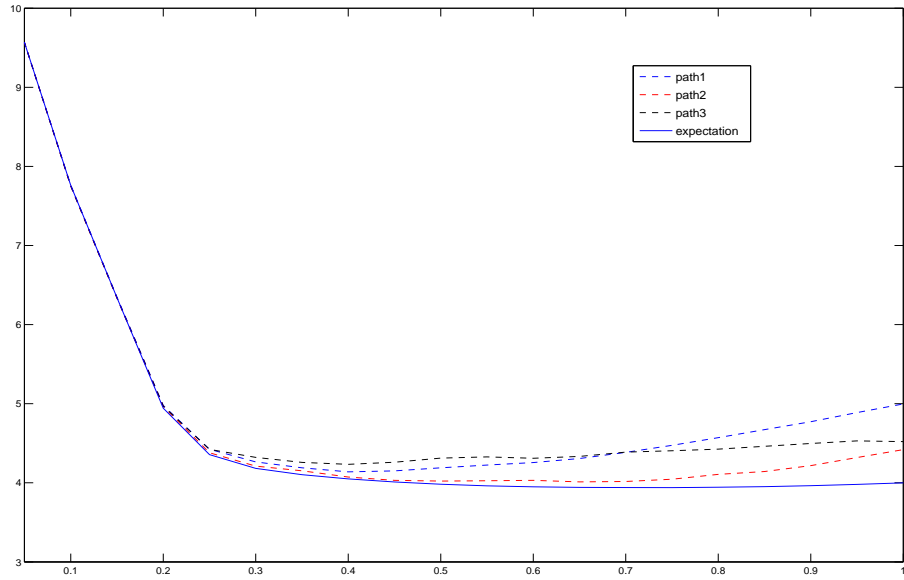


FIGURE 4. Plot of $t \mapsto \mathcal{E}(t)$, expectation and three individual paths

8. APPENDIX

For the reader's convenience we will recall the following lemmas proved in [15].

Lemma 8.1. *For any real constants λ_1 and λ_2 with $\lambda_1 \neq 0$, if $\boldsymbol{\psi}, \boldsymbol{\zeta} \in \mathbb{R}^3$ satisfy $|\boldsymbol{\zeta}| = 1$, then there exists $\boldsymbol{\varphi} \in \mathbb{R}^3$ satisfying*

$$(8.1) \quad \lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{\zeta} = \boldsymbol{\psi}.$$

As a consequence, if $\boldsymbol{\zeta} \in H^1((0, T); \mathbb{H}^1(D))$ with $|\boldsymbol{\zeta}(t, x)| = 1$ a.e. in D_T and $\boldsymbol{\psi} \in L^2((0, T); W^{1,\infty}(D))$, then $\boldsymbol{\varphi} \in L^2((0, T); \mathbb{H}^1(D))$.

Lemma 8.2. *For any $\mathbf{v} \in \mathbb{C}(D)$, $\mathbf{v}_h \in \mathbb{V}_h$ and $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D_T)$ there hold*

$$\begin{aligned} \|I_{\mathbb{V}_h} \mathbf{v}\|_{\mathbb{L}^\infty(D)} &\leq \|\mathbf{v}\|_{\mathbb{L}^\infty(D)}, \\ \|\mathbf{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\mathbf{m}_{h,k}^- \times \boldsymbol{\psi})\|_{\mathbb{L}([0,T], \mathbb{H}^1(D))}^2 &\leq ch^2 \|\mathbf{m}_{h,k}^-\|_{\mathbb{L}([0,T], \mathbb{H}^1(D))}^2 \|\boldsymbol{\psi}\|_{\mathbb{W}^{2,\infty}(D_T)}^2, \end{aligned}$$

where $\mathbf{m}_{h,k}^-$ is defined in Definition 6.1

The next lemma defines a discrete \mathbb{L}^p -norm in \mathbb{V}_h which is equivalent to the usual \mathbb{L}^p -norm.

Lemma 8.3. *There exist h -independent positive constants C_1 and C_2 such that for all $p \in [1, \infty)$ and $\mathbf{u} \in \mathbb{V}_h$ there holds*

$$C_1 \|\mathbf{u}\|_{\mathbb{L}^p(\Omega)}^p \leq h^d \sum_{n=1}^N |\mathbf{u}(\mathbf{x}_n)|^p \leq C_2 \|\mathbf{u}\|_{\mathbb{L}^p(\Omega)}^p,$$

where $\Omega \subset \mathbb{R}^d$, $d=1,2,3$.

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